Virtual Morse–Bott theory on analytic spaces, moduli spaces of SO(3) monopoles, and applications to four-manifolds

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Outline

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References

This talk is based on joint work with **Tom Leness** in our recent paper [12]:

- *Introduction to virtual Morse–Bott theory on analytic spaces, moduli spaces of* SO(3) *monopoles, and applications to four-manifolds* (with T. Leness), arXiv:2010.15789.

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Introduction

Introduction I

We shall describe a new approach to Morse–Bott theory, called *virtual Morse–Bott theory*, that applies to singular (real or complex) analytic spaces that arise in gauge theory, including moduli spaces of

- SO(3) monopoles over closed smooth four-manifolds,
- Stable holomorphic pairs of bundles and sections over closed complex Kähler surfaces,
- Higgs pairs over closed Riemann surfaces,

and potentially other non-smooth moduli spaces too.

Such moduli spaces over complex Kähler surfaces or Riemann surfaces are *complex analytic spaces* (locally equivalent to complex analytic varieties), with Kähler metrics and Hamiltonian circle actions.

Introduction II

For a smooth four-manifold that is almost Hermitian (as are four-manifolds of Seiberg–Witten simple type), where

- the almost complex structure is not necessarily integrable and
- the fundamental two-form defined by the almost complex structure and Riemannian metric is not necessarily closed,

one can still show that the moduli space of SO(3) monopoles is a *real analytic space* and (after some work) that it is almost Hermitian [13].

Such almost Hermitian, real analytic moduli spaces carry a circle action compatible with the almost complex structure and Riemannian metric and a corresponding Hamiltonian function to which a sharper version of our virtual Morse–Bott theory method applies.

We shall outline how virtual Morse–Bott theory may help prove the

Introduction III

Conjecture 1.1 (Bogomolov–Miyaoka–Yau (BMY) inequality for 4-manifolds with non-zero Seiberg–Witten invariants)

If X is a closed, oriented, smooth 4-manifold with $b_1(X) = 0$, odd $b^+(X) \geq 3$, and Seiberg–Witten simple type with a non-zero Seiberg–Witten invariant, then

$$c_1(X)^2 \leq 9\chi_h(X). \tag{1}$$

Yau [62] proved (1) for a compact Kähler surface X with ample canonical bundle using his existence of a Kähler–Einstein metric whose Ricci curvature is a negative constant [63] and a Chern–Weil inequality [59].

Simpson [54] proved (1) for such surfaces by solving the Yang–Mills–Higgs equation on a stable Higgs bundle of rank 3 over X and applying the Bogomolov–Gieseker inequality [37, 42].

Introduction IV

If X obeys the hypotheses of Conjecture 1.1, then it has an almost complex structure J [40] and in the inequality (1), which is equivalent to

$$c_1(X)^2 \leq 3c_2(X),$$

the Chern classes are those of the complex vector bundle (TX, J).

Inequality (1) was proved independently by Miyaoka [46] using methods of algebraic geometry.

Conjecture 1.1 is based on [55, Problem 4] (see also Kollár [39]), though often stated for simply connected symplectic 4-manifolds — see Gompf and Stipsicz [23, Remark 10.2.16 (c)] or Stern [55, Problem 2].

To put (1) in perspective, *any* closed, oriented, topological 4-manifold with $b_1(X) = 0$ has $c_1(X)^2 \leq 10\chi_h(X) - 1$, using $b^-(X) \geq 0$.

Frankel's Theorem for the Hamiltonian function for a circle action on a smooth almost Hermitian manifold

Frankel's Theorem for almost Hermitian manifolds I

The version of Frankel's Theorem [21, Section 3] that we prove and apply in [12] is more general because we allow for circle actions on closed, smooth manifolds (M, g, J) that are only assumed to be *almost Hermitian*, rather than (almost) Kähler. Hence,

- the almost complex structure J need not be integrable and
- the fundamental two-form $\omega = g(\cdot, J\cdot)$ defined by the compatible pair (g, J) is non-degenerate but not required to be closed,

whereas Frankel assumed in [21, Section 3] that ω was closed.

Frankel notes [21, p. 1] that the main results of his article hold when ω is a g-harmonic, symplectic form.

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Frankel's Theorem for almost Hermitian manifolds II

Recall that $J \in C^\infty(\mathrm{End}_{\mathbb{R}}(TM))$ is an *almost complex structure* on M if

$$J^2 = -\mathrm{id}_{TM}$$

and J is *orthogonal with respect to* or *compatible with* a Riemannian metric g on M if

$$g(JX, JY) = g(X, Y)$$

for all vector fields $X, Y \in C^\infty(TM)$.

Frankel's Theorem for almost Hermitian manifolds III

Theorem 1 (Frankel's Theorem for circle actions on almost Hermitian manifolds)

(Compare Frankel [21, Section 3].) Let (M, g, J) be a finite-dimensional, smooth, almost Hermitian manifold with fundamental two-form $\omega = g(\cdot, J\cdot)$. Assume that M has a smooth circle action $\rho : S^1 \times M \to M$ and let $\rho_* : S^1 \times TM \to TM$ denote the induced circle action on the tangent bundle TM given by $\rho_*(e^{i\theta})v = D_2\rho(e^{i\theta}, p)v$, for all $v \in T_pM$ and $e^{i\theta} \in S^1$. Assume that the circle action is orthogonal with respect to g and compatible with J in the sense that

$$g\left(\rho_*(e^{i\theta})v, \rho_*(e^{i\theta})w\right) = g(v, w) \quad \text{and} \quad J\rho_*(e^{i\theta})v = \rho_*(e^{i\theta})Jv, \quad \text{for all } p \in M, v, w \in T_pM, \text{ and } e^{i\theta} \in S^1.$$

Assume further that the circle action is Hamiltonian in the sense that there exists a function $f \in C^\infty(M, \mathbb{R})$ such that $df = \iota_X\omega$, where $X \in C^\infty(TM)$ is the vector field generated by the circle action, so $X_p = D_1\rho(1, p)$ for all $p \in M$ and

$$\iota_X\omega(Y) = \omega(X, Y) = g(X, JY), \quad \text{for all } Y \in C^\infty(TM).$$

Frankel's Theorem for almost Hermitian manifolds IV

Theorem 1 (Frankel's Theorem for circle actions on almost Hermitian manifolds)

If $p \in M$ is a critical point of the Hamiltonian function f, and thus a fixed point of the circle action, then

- *the eigenvalues of the Hessian $\operatorname{Hess}_g f \in \operatorname{End}(T_pM)$ of f are given by the weights of the circle action on T_pM,*
- *f is Morse–Bott at p in the sense that in a small enough open neighborhood of p, the critical set $\operatorname{Crit} f := \{q \in M : df(q) = 0\}$ is a smooth submanifold with tangent space $T_p \operatorname{Crit} f = \operatorname{Ker} \operatorname{Hess}_g f(p)$, and*
- *each connected component of $\operatorname{Crit} f$ has even dimension and even codimension in M.*

We prove Theorem 1 and further extensions in [12].

Frankel's Theorem for almost Hermitian manifolds V

Recall that we define the *gradient vector field* $\operatorname{grad}_g f \in C^\infty(TM)$ associated to any function $f \in C^\infty(M, \mathbb{R})$ by the relation $g(\operatorname{grad}_g f, Y) := df(Y)$, for all $Y \in C^\infty(TM)$.

Consequently, the Hamiltonian function f in Theorem 1 obeys $g(\operatorname{grad}_g f, Y) = g(X, JY) = -g(JX, Y)$, that is

$$\operatorname{grad}_g f = -JX.$$

If ∇^g denotes the covariant derivative for the Levi–Civita connection on TM defined by the Riemannian metric g, then one can define the Hessian of $f \in C^\infty(M, \mathbb{R})$ by

$$\operatorname{Hess}_g f := \nabla^g \operatorname{grad}_g f \in C^\infty(\operatorname{End}(TM)).$$

Frankel's Theorem for almost Hermitian manifolds VI

For a critical point $p \in M$, Theorem 1 implies that subspace $T_p^- M \subset T_pM$ on which the Hessian $\mathrm{Hess}_g f(p) \in \mathrm{End}(T_pM)$ is *negative definite* is equal to the subspace of T_pM on which the circle acts with *negative weight*.

Hence, the *Morse–Bott index* of f at a critical point p, which by definition is the dimension of the subspace $T_p^- M$, is equal to the dimension of the subspace of T_pM on which the circle acts with *negative weight*.

Virtual Morse–Bott index for the Hamiltonian function of a circle action on an analytic space

Virtual Morse–Bott index for an analytic space I

As illustrated by well-known results due to Hitchin [33, Proposition 7.1 and Theorem 7.6] for the moduli space of Higgs bundles over a Riemann surface, Theorem 1 is remarkably useful.

However, the hypotheses of Theorem 1 limit its applications to smooth manifolds and, while its application in [33] has been generalized from smooth manifolds to orbifolds (see Nasatyr and Steer [49]), those extensions do not encompass the generality allowed by Goresky and MacPherson [24] in their far-reaching development of Morse theory for *stratified spaces*.

One of our goals in our article [12] is to indicate how Frankel's Theorem 1 has useful generalizations to (real or complex) *analytic spaces* that are *not* necessarily smooth.

Virtual Morse–Bott index for an analytic space II

Analytic spaces are locally defined as the zero loci of finitely many analytic functions.

For the development of *complex* analytic spaces, we refer to Abhyankar [1], Aroca, Hironaka, and Vicente [4] (based on the earlier three-volume series by Aroca, Hironaka, and Vicente [32, 3, 2]), Fischer [20], Bierstone and Milman [5, Section 2], Grauert and Remmert [26], Griffiths and Harris, [27], or Narasimhan [48].

For the analogous development of *real* analytic spaces, we refer to Hironaka [30, 31] and Guaraldo, Macrì, and Tancredi [28].

We first have the simpler

Virtual Morse–Bott index for an analytic space III

Theorem 2 (Virtual Morse–Bott index of critical points of a real analytic function on a real analytic space)

Let X be a finite-dimensional real analytic manifold, $M \subset X$ be a real analytic subspace, $p \in M$ be a point, and $F : \mathscr{U} \to \mathbb{R}^n$ be a real analytic, local defining function for M on an open neighborhood $\mathscr{U}$ of p in the sense that $M \cap \mathscr{U} = F^{-1}(0) \cap \mathscr{U}$. Let $T_pM = \operatorname{Ker} dF(p)$ denote the Zariski tangent space to M at p. Let $f : X \to \mathbb{R}$ be a real analytic function and assume that p is a **Morse–Bott critical point** *of the restriction $f : M \to \mathbb{R}$ in the sense that*

- $\mathscr{C} = \{q \in M \cap \mathscr{U} : \operatorname{Ker} df(q) = T_qM\}$ *is a real analytic submanifold of X, and*
- $\operatorname{Ker} df(p) = T_pM$ *and* $T_p\mathscr{C} = \operatorname{Ker} \operatorname{Hess} f(p)$.

Let $\operatorname{Ker}^{\pm} dF(p) = T_p^{\pm}M \subset T_pM$ denote the maximal positive and negative real subspaces for $\operatorname{Hess} f(p) \in \operatorname{End}_{\mathbb{R}}(T_pM)$. If the **virtual Morse–Bott index**

$$\lambda_p^-(f) := \dim_{\mathbb{R}} \operatorname{Ker}^- dF(p) - \dim_{\mathbb{R}} \operatorname{Coker} dF(p), \tag{2}$$

is positive, then p is not a local minimum for $f : M \to \mathbb{R}$.

Virtual Morse–Bott index for an analytic space IV

We next have the following generalization of Theorem 1, specialization of Theorem 2, and sharpening of the definition of virtual Morse–Bott index.

Theorem 3 (Virtual Morse–Bott index of critical points of a Hamiltonian function for a circle action on a complex analytic space)

Let X be a complex, finite-dimensional, Kähler manifold with circle action that is compatible with the complex structure and induced Riemannian metric. Assume that the circle action is Hamiltonian with real analytic Hamiltonian function $f : X \to \mathbb{R}$ such that $df = \iota_\xi \omega$, where ω is the Kähler form on X and ξ is the vector field on X generated by the circle action. Let $M \subset X$ be a complex analytic subspace, $p \in M$ be a point, and $F : \mathscr{U} \to \mathbb{C}^n$ be a complex analytic, circle equivariant, local defining function for M on an open neighborhood $\mathscr{U} \subset X$ of p in the sense that $M \cap \mathscr{U} = F^{-1}(0) \cap \mathscr{U}$. Let

- $\mathbf{H}_p^2 \subset \mathbb{C}^n$ *denote the orthogonal complement of* $\operatorname{Ran} dF(p) \subset \mathbb{C}^n$,
- $\mathbf{H}_p^1 = \operatorname{Ker} dF(p) \subset T_pX$ *denote the Zariski tangent space to M at p, and*
- $M^{\mathrm{vir}} \subset X$ *denote the complex, Kähler submanifold given by $F^{-1}(\mathbf{H}_p^2) \cap \mathscr{U}$ and observe that $T_pM^{\mathrm{vir}} = \mathbf{H}_p^1$.*

Virtual Morse–Bott index for an analytic space V

Theorem 3 (Virtual Morse–Bott index of critical points of a Hamiltonian function for a circle action on a complex analytic space)

If p is a critical point of $f : M \to \mathbb{R}$ *in the sense that* $\operatorname{Ker} df(p) = \mathbf{H}_p^1$*, then p is a fixed point of the induced circle action on* M^{vir}*.*

- *Let* $S \subset M^{\mathrm{vir}}$ *be the connected component containing p of the complex analytic submanifold of* M^{vir} *given by the set of fixed points of the circle action on* M^{vir} *and assume that* $S \subset M$*.*
- *Let* $\mathbf{H}_p^{1,-} \subset \mathbf{H}_p^1$ *and* $\mathbf{H}_p^{2,-} \subset \mathbf{H}_p^2$ *denote the subspaces on which the circle acts with negative weight.*

If the **virtual Morse–Bott index**

$$\lambda_p^-(f) := \dim_{\mathbb{R}} \mathbf{H}_p^{1,-} - \dim_{\mathbb{R}} \mathbf{H}_p^{2,-} \tag{3}$$

is positive, then p is not a local minimum for $f : M \to \mathbb{R}$*.*

Virtual Morse–Bott index for an analytic space VI

Extensions and generalizations of Theorem 3

As we show in [12, 13], Theorem 3 generalizes to where X is a **real analytic, almost Hermitian manifold**.

That statement and its proof are provided in [12, 13].

Theorem 3 suffices for applications to the moduli spaces considered in [12] (including this talk) and whose top strata of smooth points are known to be complex Kähler.

Moduli spaces of anti-self-dual connections, Seiberg–Witten monopoles, and SO(3) monopoles

ASD connections, Seiberg–Witten & SO(3) monopoles I

Four-manifold topology

For a closed topological four-manifold X, we define

$$c_1(X)^2 := 2e(X) + 3\sigma(X) \quad \text{and} \quad \chi_h(X) := \frac{1}{4}(e(X) + \sigma(X)),$$

where $e(X) = 2 - 2b_1(X) + b_2(X)$ and $\sigma(X) = b^+(X) - b^-(X)$ are the Euler characteristic and signature of X, respectively.

We call X **standard** if it is closed, connected, oriented, and smooth with odd $b^+(X) \geq 3$ and $b_1(X) = 0$.

If Q_X is the intersection form on $H_2(X;\mathbb{Z})$, then $b^\pm(X)$ are the dimensions of the maximal positive and negative subspaces of Q_X on $H_2(X;\mathbb{R})$.

ASD connections, Seiberg–Witten & SO(3) monopoles II

Seiberg–Witten monopoles

For a standard 4-manifold X, its *Seiberg–Witten invariants* define a function $SW_X : \mathrm{Spin}^c(X) \ni \mathfrak{s} \mapsto SW_X(\mathfrak{s}) \in \mathbb{Z}$ on the set of spin^c structures.

A spin^c *structure* $\mathfrak{s} = (\rho, W)$ is a pair of rank-2 Hermitian vector bundles $W^\pm$ over X with $W = W^+ \oplus W^-$ and a *Clifford multiplication map*, $\rho : T^*X \to \mathrm{Hom}_{\mathbb{C}}(W^+, W^-)$.

The *Seiberg–Witten moduli space* $M_\mathfrak{s}$ is the set of gauge-equivalence classes of solutions to the *Seiberg–Witten* U(1)*-monopole equations* and is an orientable, compact, finite-dimensional, smooth manifold [47, 50, 53].

A *Seiberg–Witten invariant* $SW_X(\mathfrak{s})$ is defined by pairing a natural cohomology class with $[M_\mathfrak{s}]$ or counting points when $M_\mathfrak{s}$ is zero-dimensional.

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ASD connections, Seiberg–Witten & SO(3) monopoles III

One calls $c_1(\mathfrak{s}) := c_1(W^+) \in H^2(X;\mathbb{Z})$ a *Seiberg–Witten basic class* if $SW_X(\mathfrak{s}) \neq 0$.

The set of Seiberg–Witten basic classes, $B(X) := \{c_1(\mathfrak{s}) : SW_X(\mathfrak{s}) \neq 0\}$, is finite.

One says that X has *Seiberg–Witten simple type* if $K^2 = c_1(X)^2$ for all $K \in B(X)$.

All known standard 4-manifolds have simple type (see [41, Conjecture 1.6.2]).

ASD connections, Seiberg–Witten & SO(3) monopoles IV

Anti-self-dual connections

For $w \in H^2(X;\mathbb{Z})$ and $4\kappa \in \mathbb{Z}$, let E be a rank-2 Hermitian bundle over X with $c_1(E) = w$ and Pontrjagin number $p_1(\mathfrak{su}(E)) = -4\kappa$, where $\mathfrak{su}(E) \subset \mathfrak{gl}(E)$ is the SO(3) subbundle of trace-zero, skew-Hermitian endomorphisms of E.

Let $\mathscr{B}_E$ be the quotient of the space of fixed-determinant, unitary connections A on E by the group $\mathscr{G}_E$ of determinant-one, unitary automorphisms of E.

The *moduli space of projectively anti-self-dual (ASD) connections* on E is

$$M_\kappa^w(X) = \{[A] \in \mathscr{B}_E : (F_A^+)_0 = 0\},$$

where F_A^+ is the self-dual component defined by a metric g on X of the curvature F_A of A and $(F_A^+)_0$ is the trace-free component of F_A^+.

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ASD connections, Seiberg–Witten & SO(3) monopoles V

Then $M_\kappa^w(X)$ is an oriented smooth manifold [7] (for generic g) and non-compact due to energy bubbling, but admits an *Uhlenbeck compactification* $\bar{M}_\kappa^w(X)$ as a closed subspace of the compact space

$$IM_\kappa^w(X) := \bigsqcup_{\ell=0}^{N} (M_{\kappa-\ell}^w(X) \times \mathrm{Sym}^\ell(X))$$

of gauge-equivalence classes of *ideal ASD connections* [7, 60, 61].

ASD connections, Seiberg–Witten & SO(3) monopoles VI

SO(3) monopoles

We briefly introduce the moduli subspace $\mathscr{M}_{\mathfrak{t}} \subset \mathscr{C}_{\mathfrak{t}}$ of SO(3) monopoles, where $\mathfrak{t} = (\rho, W, E)$ is a *spin*^{u} *structure* over X and $\mathscr{C}_{\mathfrak{t}}$ is the quotient by $\mathscr{G}_E$ of the space of pairs (A, Φ) of fixed-determinant, unitary connections A on a Hermitian rank-2 vector bundle E and sections Φ of $W^+ \otimes E$.

We call (A, Φ) an SO(3) *monopole* if

$$(F_A^+)_0 - \rho^{-1}(\Phi \otimes \Phi^*)_{00} = 0 \quad \text{and} \quad D_A\Phi = 0, \tag{4}$$

where the section $(\Phi \otimes \Phi^*)_{00}$ of $\mathfrak{su}(W^+) \otimes \mathfrak{su}(E)$ is the trace-free component of $\Phi \otimes \Phi^*$ of $\mathfrak{u}(W^+) \otimes \mathfrak{u}(E)$ and D_A is the Dirac operator and $\rho : \wedge^+(T^*X) \to \mathfrak{su}(W^+)$ is an isomorphism of SO(3) bundles.

We let $\mathscr{C}_{\mathfrak{t}}^{*,0} \subset \mathscr{C}_{\mathfrak{t}}$ denote the Banach manifold of irreducible, non-zero-section pairs.

ASD connections, Seiberg–Witten & SO(3) monopoles VII

The space $\mathscr{M}_{\mathfrak{t}}$ is noncompact due to energy bubbling, but admits an Uhlenbeck compactification $\bar{\mathscr{M}}_{\mathfrak{t}}$ as a closed subspace of the compact space

$$\mathscr{I}\mathscr{M}_{\mathfrak{t}} := \bigsqcup_{\ell=0}^{N} (\mathscr{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^{\ell}(X))$$

of *ideal* SO(3) *monopoles* [14], where $\mathfrak{t}_\ell := (\rho, W, E_\ell)$ has $c_1(E_\ell) = c_1(E)$ and $c_2(E_\ell) = c_2(E) - \ell$.

ASD connections, Seiberg–Witten & SO(3) monopoles VIII

The circle action on sections Φ induces a circle action on $\bar{\mathscr{M}}_{\mathfrak{t}}$ with two types of fixed points, represented by triples $(A, \Phi, \mathbf{x})$ such that

- $\Phi \equiv 0$ or
- A is a reducible connection.

For points $[A, \Phi, \mathbf{x}] \in \bar{\mathscr{M}}_{\mathfrak{t}}$, there are bijections between

- subsets where $\Phi \equiv 0$ and the moduli space $\bar{M}_{\kappa}^{w}(X)$ of ideal ASD connections and
- subsets where A is reducible with respect to splittings, $E_{\ell} = L_1 \oplus L_2$, and Seiberg–Witten moduli spaces $M_{\mathfrak{s}}$ defined by $\mathfrak{s} = (\rho, W \otimes L_1)$.

For generic geometric perturbations, $\mathscr{M}_{\mathfrak{t}}^{*,0} = \mathscr{M}_{\mathfrak{t}} \cap \mathscr{C}_{\mathfrak{t}}^{*,0}$ is a finite-dimensional smooth manifold [14, 10, 58].

ASD connections, Seiberg–Witten & SO(3) monopoles IX

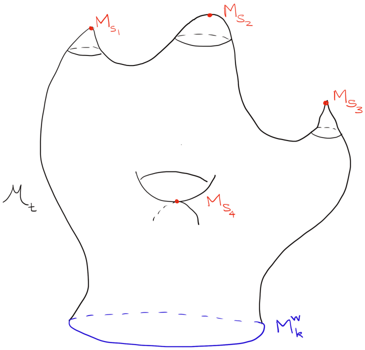

Figure 4.1: SO(3) monopole moduli space $\mathscr{M}_{\mathfrak{t}}$ with Seiberg–Witten moduli subspaces $M_{\mathfrak{s}_i}$ and moduli subspace $M^w_\kappa(X, g)$ of anti-self-dual connections

Virtual Morse–Bott theory and existence of anti-self-dual connections

Virtual Morse–Bott theory and ASD connections I

Recall [7] that the *expected dimension* of the moduli space $M_\kappa^w(X,g)$ of g-anti-self-dual connections on $\mathfrak{su}(E)$ is given by

$$\dim M_\kappa^w(X,g) = -2p_1(\mathfrak{su}(E)) - 6\chi_h(X).$$

When g is *generic* in the sense of [7, 22], then $M_\kappa^w(X,g)$ is an open, smooth manifold if non-empty.

As κ increases relative to $\chi_h(X)$, the expected dimension of $M_\kappa^w(X,g)$ increases and it becomes easier to prove that this moduli space is non-empty.

Indeed, existence results due to Taubes [56, Theorems 1.1 and 1.2] or Taylor [57, Theorem 1.1] imply that $M_\kappa^w(X,g)$ is non-empty when κ is sufficiently large relative to $\chi_h(X)$.

Virtual Morse–Bott theory and ASD connections II

By contrast, as κ becomes smaller (equivalently, as $p_1(\mathfrak{su}(E))$ becomes larger), it becomes more difficult to prove that $M_\kappa^w(X,g)$ is non-empty.

The *geography* question (see Gompf and Stipsicz [23]) asks which values of signature and Euler characteristic can be realized by smooth four-manifolds which have a given geometric structure, such as a

- complex structure,
- symplectic structure,
- an Einstein metric,
- have non-trivial Donaldson invariants,
- have non-trivial Seiberg–Witten invariants.

Virtual Morse–Bott theory and ASD connections III

We now suppose the topology of E is constrained by a *basic lower bound*,

$$p_1(\mathfrak{su}(E)) \geq c_1(X)^2 - 12\chi_h(X), \tag{5}$$

and ask whether existence of a spin^c structure $\mathfrak{s}$ over X with *non-zero Seiberg–Witten invariant* $SW_X(\mathfrak{s})$ implies that $M_\kappa^w(X,g)$ is non-empty.

If $\mathfrak{su}(E)$ also admits an *ASD connection* and the metric g on X is generic, then the moduli space $M_\kappa^w(g)$ of ASD connections on $\mathfrak{su}(E)$ with $w = c_1(E)$ and $\kappa = -\frac{1}{4}p_1(\mathfrak{su}(E))$ is non-empty, so

$$0 \leq \frac{1}{2}\dim M_\kappa^w(g) = -p_1(\mathfrak{su}(E)) - 3\chi_h(X) \leq -c_1(X)^2 + 9\chi_h(X),$$

since $M_\kappa^w(g)$ is a smooth manifold, and thus yielding the Bogomolov–Miyaoka–Yau inequality (1).

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Virtual Morse–Bott theory and ASD connections IV

Our approach to (1) hinges on proving existence of ASD connections on $\mathfrak{su}(E)$ for generic Riemannian metrics g on X via our *virtual Morse-Bott theory*.

This strategy uses our link pairing formulae [16] and the hypothesis in Conjecture 1.1 that X admits a spin^c structure $\mathfrak{s}$ with non-zero Seiberg–Witten invariant $SW_X(\mathfrak{s})$.

Remarks

Of course if $M_\kappa^w(X, g)$ has expected dimension 2δ and a Donaldson invariant of degree δ is non-zero, that would imply that $M_\kappa^w(X, g)$ is non-empty and Witten's Formula [19] expresses Donaldson invariants in terms of Seiberg–Witten invariants and so this is one way in which Seiberg–Witten invariants yield information about $M_\kappa^w(X, g)$.

Virtual Morse–Bott theory and ASD connections V

However, this route via Witten's Formula presumes that the expected dimension of $M_\kappa^w(X,g)$ is non-negative and moreover does not take into account the phenomenon of *superconformal simple type*, where Donaldson invariants of sufficiently low degree are zero even when Seiberg–Witten invariants are not all zero [11, 18, 44, 45].

Our approach to proving that $M_\kappa^w(X,g)$ is non-empty for standard four-manifolds X of Seiberg–Witten simple type under the constraint (5) relies on *virtual Morse–Bott theory* on the moduli space of SO(3) monopoles.

Virtual Morse–Bott theory on the moduli space of SO(3) monopoles over a closed, complex Kähler surface

Virtual Morse–Bott theory on moduli spaces I

Our strategy to use compactifications of the moduli space $\mathscr{M}_{\mathfrak{t}}$ of SO(3) monopoles to prove existence of an anti-self-dual connection on $\mathfrak{su}(E)$ satisfying (5) is broadly modeled on that of Hitchin [33].

We first prove existence of a spin^u structure $\mathfrak{t} = (\rho, W, E)$ with $p_1(\mathfrak{su}(E))$ satisfying (5) and non-empty SO(3) monopole moduli space $\mathscr{M}_{\mathfrak{t}}$.

We then show that for Kähler surfaces X, all critical points of Hitchin's function on $\mathscr{M}_{\mathfrak{t}}$ are given by

- points in $M_{\kappa}^{w}(X, g) \subset \mathscr{M}_{\mathfrak{t}}$, or
- points in moduli subspaces $M_{\mathfrak{s}} \subset \mathscr{M}_{\mathfrak{t}}$, defined by spin^c structures $\mathfrak{s} = (\rho, W)$, of Seiberg–Witten monopoles with positive virtual Morse–Bott index.

Virtual Morse–Bott theory on moduli spaces II

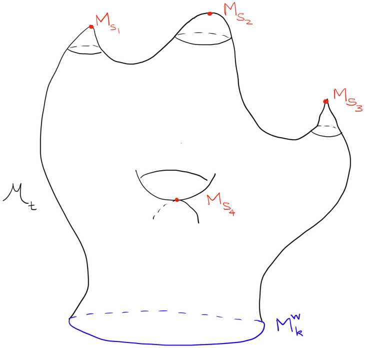

Figure 6.2: SO(3) monopole moduli space $\mathscr{M}_{\mathfrak{t}}$ with Seiberg–Witten moduli subspaces $M_{\mathfrak{s}_i}$ and moduli subspace $M_\kappa^w(X, g)$ of anti-self-dual connections

Virtual Morse–Bott theory on moduli spaces III

Unlike the case of Higgs pairs in [33], we must address two fundamental new difficulties in our application that do not arise in [33]:

1. The strata $M_{\mathfrak{s}} \subset \mathscr{M}_{\mathfrak{t}}$ of moduli subspaces of Seiberg–Witten monopoles are smooth submanifolds but **not necessarily smoothly embedded as submanifolds** of $\mathscr{M}_{\mathfrak{t}}$; and
2. The moduli space $\mathscr{M}_{\mathfrak{t}}$ of SO(3) monopoles is **non-compact due to energy bubbling**.

Hitchin [33] assumes that the degree of the Hermitian vector bundle E is odd and that its rank is two.

More generally, it is known (see Fan [9] or Gothen [25] and references therein) that if the degree and rank of the Hermitian vector bundle E in the equations (5.1) for a Higgs pair are *coprime*, then the linearization of

Virtual Morse–Bott theory on moduli spaces IV

the equations (5.1) at a Higgs pair (A, Φ) has *vanishing cokernel* and Higgs pairs (A, Φ) have trivial isotropy subgroups in Aut E.

However, when the degree and rank of E are not coprime, one may encounter problems similar to those in Item (1), where the linearization of the SO(3) monopole equations at a Seiberg–Witten pair (A, Φ) may have a non-vanishing cokernel.

The virtual Morse–Bott theory method that we introduce in [12] is designed to address such difficulties:

We replace the **classical** Morse–Bott index [6] of a smoothly embedded critical submanifold with the **virtual** Morse–Bott index of a smooth critical manifold that need not be smoothly embedded as a submanifold.

Virtual Morse–Bott theory on moduli spaces V

Hitchin's Morse function

We consider the following analogue of Hitchin's Morse function [33],

$$f : \mathscr{I}\mathscr{M}_{\mathfrak{t}} \ni [A, \Phi, \mathbf{x}] \mapsto f[A, \Phi, \mathbf{x}] = \frac{1}{2}\|\Phi\|^2_{L^2(X)} \in \mathbb{R} \tag{6}$$

on the compact, smoothly stratified space of ideal SO(3) monopoles $\mathscr{I}\mathscr{M}_{\mathfrak{t}}$ containing the *Uhlenbeck compactification* $\bar{\mathscr{M}}_{\mathfrak{t}}$ of $\mathscr{M}_{\mathfrak{t}}$ (see [14]).

The function f is continuous on $\mathscr{I}\mathscr{M}_{\mathfrak{t}}$ (see [14] or [17]) and smooth, but not necessarily a Morse–Bott function on smooth strata of $\mathscr{I}\mathscr{M}_{\mathfrak{t}}$.

We say that an interior point $p = [A, \Phi] \in \mathscr{M}_{\mathfrak{t}}$ is a *critical point* of f if $df(p)$ is zero on the Zariski tangent space $T_p\mathscr{M}_{\mathfrak{t}}$; this definition can be extended to points in the ideal boundary $\partial\mathscr{M}_{\mathfrak{t}}$.

Virtual Morse–Bott theory on moduli spaces VI

Feasibility of the SO(3)-monopole cobordism method

The first step in our program is to construct a spin^u structure $\mathfrak{t} = (\rho, W, E)$ with $p_1(\mathfrak{su}(E))$ satisfying (5) and for which $\mathscr{M}_{\mathfrak{t}}^{*,0}$ is non-empty so that the gradient flow of (1.4) will have a starting point.

To obtain greater control over the characteristic classes of the spin^u structure, we work on the blow-up $\widetilde{X} := X \# \overline{\mathbb{CP}}^2$ of X.

Because $c_1(\widetilde{X})^2 = c_1(X)^2 - 1$, we replace (5) with the condition that $p_1 \geq c_1(\widetilde{X}) + 1 - 12\chi_h(\widetilde{X})$. We then prove

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Theorem 4 (Feasibility of spin^u structures and positivity of virtual Morse–Bott indices)

Let X be a standard four-manifold of Seiberg–Witten simple type, let $\widetilde{X} = X \# \overline{\mathbb{CP}}^2$ denote the blow-up of X, and let $\tilde{\mathfrak{s}}$ be a spin^c structure over X with non-zero Seiberg–Witten invariant $SW_{\widetilde{X}}(\tilde{\mathfrak{s}})$. Then there exists a spin^u structure $\tilde{\mathfrak{t}}$ over $\widetilde{X}$ such the following hold:

1. $M_{\tilde{\mathfrak{s}}} \subset \mathscr{M}_{\tilde{\mathfrak{t}}}$;
2. *The moduli space* $\mathscr{M}_{\tilde{\mathfrak{t}}}^{*,0}$ *is non-empty; and*
3. $p_1(\tilde{\mathfrak{t}}) \geq c_1(\widetilde{X})^2 + 1 - 12\chi_h(\widetilde{X})$.

Moreover, for all non-empty Seiberg–Witten moduli subspaces $M_{\mathfrak{s}'} \subset \mathscr{M}_{\tilde{\mathfrak{t}}}$, the virtual Morse–Bott index of the Hitchin function f in (6) along $M_{\mathfrak{s}'}$ is **positive**.

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While we prove Theorem 6 in [12] for closed complex Kähler surfaces, we see no obstacle to extending it to standard four-manifolds.

Indeed, the cohomology calculations used to prove Theorem 4 do not rely on X being complex or Kähler and the corresponding elliptic deformation complexes obtained when closed complex Kähler surfaces are replaced by standard four-manifolds are perturbations by compact operators of the elliptic deformation complexes obtained for closed complex Kähler surfaces.

Virtual Morse–Bott theory on moduli spaces IX

Critical points of Hitchin's function in the moduli space of SO(3) monopoles over closed complex Kähler surfaces

The next step in our program is to use Frankel's Theorem (see Theorem 1) to identify the critical points of Hitchin's function on $\mathscr{M}_{\mathfrak{t}}$.

In this talk, we shall restrict our discussion to SO(3) monopoles over closed complex Kähler surfaces.

Suppose that (X, g, J) is a complex Kähler surface, with Kähler form $\omega(\cdot, \cdot) = g(\cdot, J\cdot)$.

A version of the Hitchin–Kobayashi bijection then identifies the moduli space $\mathscr{M}_{\mathfrak{t}}$ of SO(3) monopoles with the moduli space of stable holomorphic pairs (see Dowker [8], Lübke and Teleman [42, 43], and Okonek and Teleman [51, 52]).

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Thus $\mathscr{M}_\mathfrak{t}$ is a complex analytic space and the open subspace $\mathscr{M}_\mathfrak{t}^{\mathrm{sm}} \subset \mathscr{M}_\mathfrak{t}$ of smooth points is a complex manifold, generalizing results by Itoh [34, 35] for the complex structure of the moduli space $M_\kappa^w(X,g)$ of anti-self-dual connections via its identification with the moduli space of stable holomorphic bundles.

In [13], we extend the proofs by Itoh [36] and Kobayashi [38] of their results for $M_\kappa^w(X,g)$ to prove that the L^2 metric $\mathbf{g}$ and integrable almost complex structure $\mathbf{J}$ on $\mathscr{M}_\mathfrak{t}^{\mathrm{sm}}$ define a Kähler form $\boldsymbol{\omega} = \mathbf{g}(\cdot, \mathbf{J}\cdot)$ on $\mathscr{M}_\mathfrak{t}^{\mathrm{sm}}$.

When g (and other geometric perturbation parameters in the SO(3) monopole equations) are *generic*, then $\mathscr{M}_\mathfrak{t}^{\mathrm{sm}} = \mathscr{M}_\mathfrak{t}^{*,0}$ but if g is Kähler and thus non-generic, this equality need not hold.

We address this issue in [13] and outline them in [12].

Virtual Morse–Bott theory on moduli spaces XI

Circle actions, moment maps, Frankel's Theorem, Morse–Bott property

As in the analysis [33, Sections 6 and 7] by Hitchin, the function f in (6) is a *moment map* for the *circle action* on $\mathscr{M}_{\mathfrak{t}}^{\mathrm{sm}}$, that is,

$$df = \iota_{\xi}\omega \quad \text{on } \mathscr{M}_{\mathfrak{t}}^{\mathrm{sm}},$$

where the vector field ξ on $\mathscr{M}_{\mathfrak{t}}^{\mathrm{sm}}$ is the generator of the S^1 action on $\mathscr{M}_{\mathfrak{t}}$ given by scalar multiplication on the sections Φ.

Because the fundamental 2-form ω is *non-degenerate*, $[A, \Phi] \in \mathscr{M}_{\mathfrak{t}}$ is a critical point if and only if it is a fixed point of the S^1 action.

From our previous work on SO(3) monopoles [14], we know that $[A, \Phi]$ is a fixed point if and only if (A, Φ) is a *reducible pair* with $\Phi \not\equiv 0$ (equivalent to a *Seiberg–Witten monopole*) or $\Phi \equiv 0$ (equivalent to a *projectively anti-self-dual connection*).

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Indeed, as a consequence of Theorem 1 and our results in [14, 15] that identify the fixed points of this S^1 action, we have the

Theorem 5 (All critical points of Hitchin's Hamiltonian function represent either Seiberg–Witten monopoles or anti-self-dual connections)

Let $[A, \Phi] \in \mathscr{M}_{\mathfrak{t}}$ *be a critical point of Hitchin's function* (6). *If* $\Phi \not\equiv 0$, *then there exists a spin*^{c} *structure* $\mathfrak{s}$ *over* X *such that* $[A, \Phi] \in M_{\mathfrak{s}} \subset \mathscr{M}_{\mathfrak{t}}$.

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Virtual Morse–Bott properties

If a Seiberg–Witten fixed point $p = [A, \Phi]$ is a **smooth** point of $\mathscr{M}_{\mathfrak{t}}$ then, by arguments generalizing those of Hitchin [33, Section 7], one can apply Frankel's Theorem 1 to

- prove that f is Morse–Bott at p and
- compute the Morse index of f (the dimension of the maximal negative definite subspace of Hess $f(p)$ on $T_p\mathscr{M}_{\mathfrak{t}}$) as the dimension of the negative weight space $T_p^-\mathscr{M}_{\mathfrak{t}}$ for the S^1 action on $T_p\mathscr{M}_{\mathfrak{t}}$.

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The dimension of $T_p^- \mathscr{M}_\mathfrak{t}$ can be computed via the Atiyah–Singer Index Theorem or the Hirzebruch–Riemann–Roch Index Theorem via the identification of $\mathscr{M}_\mathfrak{t}$ with a moduli space of stable holomorphic pairs.

In [33, Proposition 7.1], Hitchin computes the Morse indices of the reducible Higgs pair points; they are always **positive**, so these points cannot be local minima.

If the Seiberg–Witten fixed point $p = [A, \Phi]$ is a **singular** point of $\mathscr{M}_\mathfrak{t}$, as is more typical, we use the fact that $M_\mathfrak{s} \subset \mathscr{M}_\mathfrak{t}$ is a submanifold of a smooth virtual moduli space $\mathscr{M}_\mathfrak{t}^{\mathrm{vir}} \subset \mathscr{C}_\mathfrak{t}$ implied by the Kuranishi model given by the elliptic deformation complex for the SO(3) monopole equations with cohomology groups $\mathbf{H}^\bullet_{A,\Phi}$.

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The space $\mathscr{M}_{\mathfrak{t}}^{\mathrm{vir}}$ is a complex Kähler manifold of dimension equal to that of the Zariski tangent space $T_p\mathscr{M}_{\mathfrak{t}}$ and contains $\mathscr{M}_{\mathfrak{t}}^{\mathrm{sm}}$ and $M_{\mathfrak{s}}$ as complex Kähler submanifolds.

The set of fixed points of the S^1 action on $\mathscr{M}_{\mathfrak{t}}^{\mathrm{vir}}$ coincides with $M_{\mathfrak{s}}$ and f is Morse–Bott on $\mathscr{M}_{\mathfrak{t}}^{\mathrm{vir}}$ with critical submanifold $M_{\mathfrak{s}}$.

(In the simpler setting of Hitchin [33, Section 7], the critical sets are smooth submanifolds of the moduli space of Higgs pairs and f is Morse–Bott.)

Here, $M_{\mathfrak{s}}$ is a smooth manifold, given by a moduli space of Seiberg–Witten monopoles that may be zero or positive-dimensional (in the latter case $SW_X(\mathfrak{s}) = 0$ when X has Seiberg–Witten simple type), but **not necessarily an embedded smooth submanifold** of $\mathscr{M}_{\mathfrak{t}}$.

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The elliptic deformation complex defining $\mathbf{H}^\bullet_{A,\Phi}$ splits into **normal** and **tangential** elliptic deformation subcomplexes with cohomology groups $\mathbf{H}^{\bullet,t}_{A,\Phi}$ and $\mathbf{H}^{\bullet,n}_{A,\Phi}$, respectively.

The normal elliptic deformation complex defining $\mathbf{H}^{\bullet,n}_{A,\Phi}$ further splits into **positive** and **negative weight** elliptic deformation subcomplexes with cohomology groups $\mathbf{H}^{\bullet,\pm}_{A,\Phi}$.

We define the **virtual Morse–Bott index** of f at p to be minus the (Atiyah–Singer) index of the **negative weight, normal subcomplex** defining $\mathbf{H}^{\bullet,-}_{A,\Phi}$ and compute these indices using the Hirzebruch–Riemann–Roch Index Theorem to give

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Virtual Morse–Bott theory on moduli spaces XVII

Theorem 6 (Virtual Morse–Bott index of Hitchin's Hamiltonian function at a point represented by a Seiberg–Witten monopole)

Let X be a closed complex Kähler surface and $[A,\Phi] \in M_{\mathfrak{s}} \subset \mathscr{M}_{\mathfrak{t}}$ be a Seiberg–Witten monopole in the SO(3) *monopole moduli space $\mathscr{M}_{\mathfrak{t}}$. Then the virtual Morse–Bott index of Hitchin's function f in* (6) *at the point $[A,\Phi]$ is given by*

$$\begin{aligned}\lambda_{[A,\Phi]}(f) &= \dim \mathbf{H}^{1,n,-}_{A,\Phi} - \dim \mathbf{H}^{2,n,-}_{A,\Phi} \dim \mathbf{H}^{0,n,-}_{A,\Phi} \\ &= -\chi_h(X) + \frac{1}{2}(c_1(\mathfrak{s}) - c_1(\mathfrak{t})) \cdot K_X - \frac{1}{2}(c_1(\mathfrak{s}) - c_1(\mathfrak{t}))^2, \end{aligned} \tag{7}$$

where $K_X \in H^2(X;\mathbb{Z})$ denotes the canonical class of X, and $c_1(\mathfrak{s}) := c_1(W^+) \in H^2(X;\mathbb{Z})$ for $W = W^+ \oplus W$, and $c_1(\mathfrak{t}) := c_1(E) \in H^2(X;\mathbb{Z})$.

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Virtual Morse–Bott theory on moduli spaces XVIII

We included $\mathbf{H}^0_{A,\Phi}$ above for completeness, but for SO(3) monopoles we have $\mathbf{H}^0_{A,\Phi} = 0$ unless $\Phi \equiv 0$ and A is reducible (a case that can be excluded by standard techniques for the moduli space of SO(3) monopoles.

For reasons explained in Remark 1.2, we expect the formula (7) for the virtual Morse–Bott index in the conclusion of Theorem 6 to continue to hold for standard four-manifolds.

Knowledge of the dimension $\dim M_{\mathfrak{s}}$ of the critical submanifold $M_{\mathfrak{s}}$ is not required to compute the index of the normal elliptic deformation subcomplexes.

If p is a **smooth** point, its virtual Morse–Bott index is equal to the **classical** Morse–Bott index.

Virtual Morse–Bott theory on moduli spaces XIX

One can use

- the Embedded Resolution of Singularities Theorem for (real or complex) analytic spaces (see Hironaka [29]), and
- generic perturbation and transversality arguments

to prove that if the **virtual Morse–Bott index** of a singular critical point p is **positive**, then p **cannot be a local minium** (see Theorem 3).

Thank you for your attention!

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