Algebraic and Geometric classification of Yang-Mills-Higgs flow lines

1. Motivation (Morse Theory)

Morse theory aims to relate the topology of a manifold $M$ to analytic properties of "well-behaved" smooth functions $f: M \rightarrow \mathbb{R}$.

Basic Example:

![Diagram of a manifold with critical points labeled $P_1$, $P_2$, $P_3$, and $P_4$.]

This is enough information to determine Morse inequalities:

$$b_k = \dim_{\mathbb{R}} H^k(M, \mathbb{C}), \quad c_k = \# \text{crit. points of index } k$$

$$P_k(M) = \sum_{i=0}^{\infty} b_i \cdot t^k \quad \quad M_k(f) = \sum_{i=0}^{\infty} c_i \cdot t^k$$

(Poincaré polynomial) \hspace{1cm} (Morse polynomial)

Then:

$$M_k(f) - P_k(M) = (1+t) \cdot R_k(t)$$

A polynomial with non-negative coefficients.

In particular, $c_k \geq b_k$ for all $k$ (Weak Morse inequalities).

We would like to use analysis to determine more information about the topology of $M$, in particular:

(a) $P_0(M)$
(b) cup product structure on $H^*(M)$.

We can do this if we know more information about spaces of gradient flow lines connecting critical points, which requires a Riemannian metric on $M$.

For the Yang-Mills flow and Yang-Mills-Higgs flow these spaces have a very interesting algebraic and geometric structure.
2. Holomorphic and Higgs bundles over Riemann surfaces

Let \( X \) be a compact Riemann surface, \( \mathcal{E} \to X \) a smooth \( r \)-vector bundle (rank \( r \), degree \( d \)).

Holomorphic structure on \( \mathcal{E} \Rightarrow \) Holomorphic transition functions

The \( \mathcal{E} \to X \) implication is well-defined operator \( \mathcal{E} \to \Omega^1(X) \otimes \mathcal{E} \).

\( H^0(\mathcal{E}) \) is the space of holomorphic sections of \( \mathcal{E} \).

For any bundle associated to \( \mathcal{E} \) (e.g., \( \text{End} \mathcal{E} \) is a bundle of endomorphisms of \( \mathcal{E} \)), there is an associated \( \mathcal{D} \)-operator

\[
\mathcal{D}_e = \partial + \text{ad} \, \mathcal{D}_e
\]

Gauge group: \( \text{Aut}(\mathcal{E}) = \text{smooth sections of the frame bundle} \)

Gauge action

\[
g \cdot (e) = \tilde{g} e g^{-1}
\]

The space of all such holomorphic structures is an affine space

\[
\mathcal{A} = \mathcal{D}_e + \Omega^1(X) \otimes \mathcal{E}
\]

A Higgs bundle \((\mathcal{E}, \rho, \sigma, \alpha)\) is a pair \((\mathcal{E}, \rho)\) where \( \rho \in \text{H}^0(\text{End} \, \mathcal{E} \otimes \mathcal{O}_X) \) (\( \mathcal{O}_X \) denotes the structure sheaf of \( X \)).

The space of all Higgs bundles is

\[
\mathcal{B} = \mathcal{A} \setminus \mathcal{D}_e \mathcal{A}
\]

The forgetful map \( \mathcal{B} \to \mathcal{A} \)

\[
\mathcal{B} \to \mathcal{A} \text{ (dimension depends on } \mathcal{D}_e \text{)}
\]

We would like to construct a moduli space of holomorphic/Higgs bundles up to gauge equivalences.

\( \mathcal{A} \) does not have good properties (e.g., is not Hausdorff).

Instead we restrict to an open subset of stable/semistable bundles.

\((\mathcal{E}, \rho)\) is

- stable if \( \deg \rho \geq \deg \rho / \text{rank } \rho \) for every proper non-zero subbundle \( \mathcal{F} \subset \mathcal{E} \);
- semistable if \( \deg \rho / \text{rank } \rho \) is invariant holomorphic subbundle \( \mathcal{F} \subset \mathcal{E} \).

Then define the moduli space of stable (resp. semistable) Higgs bundles by

\[
\mathcal{M}^s, \mathcal{M} \overset{\text{stable}}{\Rightarrow} \mathcal{M} \overset{\text{semistable}}{\Rightarrow} \mathcal{A}
\]

All the objects defined above are holomorphic.

In order to define curvature Yang-Mills-Higgs energy etc., we need to fix a Hermitian metric on \( \mathcal{E} \).
For a Hamiltonian metric in an F.

Then each holomorphic structure I has an associated Chern connection 
\( \nabla \), compatible with the metric, such that the following diagram commutes:

\[
\begin{array}{ccc}
\nabla & \rightarrow & \text{holomorphic part} \\
\downarrow & & \downarrow \\
\phi & \rightarrow & I
\end{array}
\]

Let \( F_i \) be the curvature of the
\( F \rightarrow \mathfrak{g} \otimes \mathfrak{g} \) Lie algebra group

\( \text{for} \mathfrak{g} = \mathbb{R}, \mathfrak{M} = \mathfrak{g} \otimes \mathfrak{g} \).

The Yang-Mills energy

\[
\frac{\text{YM}}{\text{Y-M}} \text{ Y-M energy}
\]

The energy minimizing connection satisfy

\[
F_i = 0 \quad \text{Y-M energy}
\]

The critical points are direct sums of harmonic sections, namely

\[
\text{critical points are direct sums of harmonic sections and }
F_i = 0
\]

We can think of these energy functions as height functions for the

\( \mathfrak{g} \)-valued maps to the fibers of the tangent bundle.

The Yang-Mills equations.

\[
\begin{align*}
\frac{\delta E_{\text{YM}}}{\delta F_i} & = 0 \\
\frac{\delta E_{\text{YM}}}{\delta A_j} & = 0
\end{align*}
\]

It is useful to study the Yang-Mills gradient flow of \( \text{YM} \) as a

\[
\frac{\delta E_{\text{YM}}}{\delta F_i} = -i \nabla^\mathfrak{g} (F_i - i \mathfrak{g}^\mathfrak{g})
\]

\[
\frac{\delta E_{\text{YM}}}{\delta A_j} = -i [A_j, F_i] + i [F_i, A_j]
\]

If \( X \) is a flow, then a solution (for \( t \to \infty \)) is generated by

\[
X(0) \to X(t)
\]

where

\[
\frac{\delta E_{\text{YM}}}{\delta F_i} = -i \nabla^\mathfrak{g} (F_i - i \mathfrak{g}^\mathfrak{g})
\]

We can also think of the flow in terms of changing the metric

\[
\nabla \rightarrow \nabla + \nabla^\mathfrak{g}
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For example, consider the flow may not exist

\[
\nabla \rightarrow \nabla + \nabla^\mathfrak{g}
\]

We need to work with the Yang-Mills equation.
1. Results about the downslopes flow

The Yang-Mills/Yang-Mills-Higgs heat flow is well studied for $t > 0$

1.1 Long-time existence (Donaldson '85, Simpson '88)

**Key idea**: Distance-decreasing property

**Initial conditions**: $(g_0, h_0)$ and $q \cdot (g_0, h_0) \neq 0$

The two solutions of the flow are $(g(t), h(t))$ and $g \cdot (g(t), h(t))$

Let $k = g \cdot g$, and define $\sigma(k) = tr(k) + tr((k)^{-1}) - 2 \log k$

Then, $(\frac{d}{dt} + i) \sigma(k) < 0$

**Distance in space of metrics**

1.2 Convergence (Ricci '92, W. '88)

$\lim_{t \to \infty} (g(t), h(t))$ exists and is a critical point

1.3 Identification of the limit (Donaldson '92, W. '88)

Every holomorphic/higgs bundle admits a

"Hitchin-Narasimhan--Seshadri" double filtration which

measures the failure of the bundle to be stable

**Theorem**: (Donaldson '92, W. '88)

The limit is isomorphic to the graded object of the

Hitchin-Narasimhan--Seshadri filtration of the initial condition

**Example**: rank $E = 2$, $L, \mathcal{E}$ is the line subbundle of maximal degree

$\Rightarrow$ H-N-S filtration is $0 \subset L_1 \subset \mathcal{E}$

$0 \to L_2 \to \mathcal{E} \to 0$

$E$ stable

$B$ pair

What about flow lines?

**Example**: $E \oplus L_1$

1. What are the conditions on $E, L_1, L_1'$

for a flow line to connect the critical points

$L, L_1, L_1'$ and $L, L_1, L_1'$?

2. What does the space of flow lines look like?

2. What about broken/unbroken flow lines?
The reverse flow near a critical point.

\[ \frac{\partial}{\partial t} \phi = \text{deg}_{L} \]

Extensions:

\[ 0 \rightarrow L_{2} \rightarrow E \rightarrow L_{1} \rightarrow 0 \]

- Linearized unstable manifold

Holomorphic structure \( (\frac{\partial}{\partial z}, 0) \), \( \alpha \in H^{0}(L_{1}, L_{2}) \subseteq H^{0}(L_{1}, \mathcal{O}) \)

Act by the gauge transformation:

\[ q_{t} = \left( e^{t \lambda}, 0 \right) \]

\[ \phi_{t}(\frac{\partial}{\partial z}, 0)q_{t} = \left( e^{t \lambda}, 0 \right) \rightarrow \left( \frac{\partial}{\partial z}, 0 \right) \text{ as } t \rightarrow -\infty \]

Now flow down using the nonlinear heat flow.

Theorem (W.) (i) This process converges as \( t \rightarrow -\infty \).

(ii) \( x_{0} = \lim_{t \rightarrow -\infty} \phi_{t} \cdot x \in E^{0} \cdot x \)

(iii) One can construct a solution to the reverse heat flow from \( x_{0} \) up to the critical point \( 0 \).

(iv) Conversely, every solution to the reverse heat flow that converges to \( L_{0} \) can be constructed in this way.

More generally, (arbitrary rank also works for Higgs bundles) a point on a flow line between two critical points \( x_{-} \) and \( x_{+} \) must admit two filtrations.

(i) H-N-S filtration whose graded object is isomorphic to \( x_{+} \)

(ii) A filtration whose graded object is a direct sum of stable bundles with increasing slope = \( \text{deg} / \text{rank} \).
(a) Hecke correspondence via space of flow lines

\[ E_0 \to E \to E_\infty \to E \to E_0 \to 0 \]

\[ E_0 \to E \to E_\infty \to E_0 \to 0 \]

Then \( E_\infty \) is a subobject of \( E_0 \)

\[ O \to E_0 \to E_\infty \to E_0 \to O \]

\[ \mu = \text{points that flow up to } C_\infty \text{ and down to } C_0 \]

\[ \text{lower crit. set} \rightarrow \text{upper crit. set} \]

\[ \text{intermediate crit. set} \rightarrow \text{critical points connected by a flow line modulo gauge} \]

Theorem (W): \( \Phi_b \) is the Hecke correspondence.

(b) Geometric classification of broken/unbroken flow lines

Fix an upper critical point \( x_\infty \in E_\infty \). Projective embedding \( X \hookrightarrow \mathbb{P} \mathbb{H} X(L, L_b) \) - general extension from before

\[ \text{intermediate crit. set} \rightarrow \text{upper crit. set} \rightarrow \text{lower crit. set} \]

\[ \text{higher rank Yang-Mills: } \text{use secant varieties of } \mathbb{P} E \subset \mathbb{P} \mathbb{H} \]

Higgs bundles: Need to take the Higgs field into account