MORSE THEORY FOR THE SPACE OF HIGGS G–BUNDLES

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ABSTRACT. Fix a C^∞ principal G–bundle E_G^0 on a compact connected Riemann surface X, where G is a connected complex reductive linear algebraic group. We consider the gradient flow of the Yang–Mills–Higgs functional on the cotangent bundle of the space of all smooth connections on E_G^0. We prove that this flow preserves the subset of Higgs G–bundles, and, furthermore, the flow emanating from any point of this subset has a limit. Given a Higgs G–bundle, we identify the limit point of the integral curve passing through it. These generalize the results of the second named author on Higgs vector bundles.

1. INTRODUCTION

The equivariant Morse theory of the Yang–Mills functional for vector bundles over a compact Riemann surface has been an extremely useful tool in studying the topology of the moduli space of semistable holomorphic bundles, beginning with the work of Atiyah and Bott in [2], and continuing with the results of Kirwan on the intersection cohomology in [15]. In the case where the rank and degree of the bundle are coprime, this program was continued further by Kirwan in [16], by Jeffrey and Kirwan (who computed the intersection pairings in [14]), and Earl and Kirwan in [11] who wrote down the relations in the cohomology ring. As a result, via Morse theory we now have a complete description of the cohomology of this space.

The convergence properties of the gradient flow of the Yang–Mills functional were first studied by Daskalopoulos in [7] and Råde in [17]. Råde studies the more general problem of the Yang–Mills flow on the space of connections on a 2 or 3 dimensional manifold, and shows that the gradient flow converges to a critical point of the Yang–Mills functional. When the base manifold is a compact Riemann surface, then Råde's results show that there exists a Morse stratification of the space of holomorphic bundles. In [7], Daskalopoulos shows that this Morse stratification of the space of holomorphic bundles coincides with the algebraically defined Harder–Narasimhan stratification used by Atiyah and Bott, and uses this to obtain information about the homotopy type of the space of strictly stable rank 2 bundles.

The analytically more complicated case of the Yang–Mills flow on the space of holomorphic structures on a Kähler surface (where bubbling occurs at isolated points on the surface in the limit of the flow) was studied by Daskalopoulos and Wentworth in [9], who showed that the algebraic and analytic stratifications coincide, and that the bubbling in the limit of the flow is determined by the algebraic properties of the initial conditions.

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The main theorem of [19] extend the gradient flow results of Daskalopoulos and Råde to the space of Higgs vector bundles over a compact Riemann surface, where the functional in question is now the Yang–Mills–Higgs functional (see the definition below). These results were then used in [8] to carry out an analog of Atiyah and Bott's construction on the space of rank 2 Higgs bundles, the first step in carrying out the Atiyah/Bott/Kirwan program described above for Higgs bundles. The purpose of the current paper is to generalize the gradient flow convergence theorem of [19] to the space of Higgs principal bundles over a compact Riemann surface.

The main result of the paper can be stated as follows. Let X be a compact Riemann surface. Given a C^∞ vector bundle V on X, the space of C^∞ forms of type $(p\,,q)$ with values in V will be denoted by $A^{p,q}(V)$. Let G a connected reductive linear algebraic group over $\mathbb{C}$, and fix a maximal compact subgroup K. Fix a principal K–bundle E_K^0 with compact structure group K, and let E_G^0 denote the associated principal bundle obtained by extending the structure group to G. Let $\mathcal{A}_0$ be the space of holomorphic structures on E_G^0, and consider the space $\mathcal{B}_0 = \mathcal{A}_0 \times A^{1,0}(\mathrm{ad}(E_G^0))$ (we show in Section 2 that this is the total space of the cotangent bundle of $\mathcal{A}_0$). A pair $(\overline{\partial}_{E_G^0}\,, \theta) \in \mathcal{B}_0$ is called a *Higgs pair* if θ is holomorphic with respect to $\overline{\partial}_{E_G^0}$. The space of Higgs pairs is denoted $\mathcal{S}(E_G^0)$, and the *Yang–Mills–Higgs functional* on the space $\mathcal{B}_0$ is given by

$$\mathrm{YMH}_G : \mathcal{B}_0 \to \mathbb{R}$$
$$\mathrm{YMH}_G(\overline{\partial}_{E_G^0}, \theta) = \left\| K(\overline{\partial}_{E_G^0}) + [\theta, \theta^*] \right\|^2 ,$$

where $K(\overline{\partial}_{E_G^0})$ is the curvature of the connection on E_K^0 associated to the holomorphic structure $\overline{\partial}_{E_G^0}$ on E_G^0 (see Section 2 and the definition in equation (2.23) for full details of this construction).

The notion of the Harder–Narasimhan reduction of a Higgs principal bundle is recalled in Section 4, and we show that the definition of socle reduction for semistable vector bundles (cf. [13]) extends to the semistable Higgs G–bundles. Combining the Harder–Narasimhan reduction with the socle reduction gives the principal bundle analog of the graded object of the Harder–Narasimhan–Seshadri filtration studied in [19]. The main theorem of this paper generalizes the results of [19] to Higgs principal bundles.

Theorem 1.1. *The gradient flow of* YMH_G *with initial conditions* $(\overline{\partial}_{E_G^0}, \theta)$ *in the space of Higgs pairs on* E_G^0 *converges to a Higgs pair isomorphic to the pair obtained by combining the socle reduction with the Harder–Narasimhan reduction of* $(\overline{\partial}_{E_G^0}, \theta)$.

The idea of the proof is to reduce to the case of Higgs vector bundles studied in [19]. Fix a C^∞ principal G–bundle E_G^0 on a compact Riemann surface X. Given a faithful representation $\rho : G \longrightarrow GL(V)$, let $\mathcal{S}(W)$ denote the space of Higgs pairs on the Hermitian vector bundle $W = E_G^0(V)$ associated to E_G^0 via ρ. We show that the Yang–Mills–Higgs flow on $\mathcal{S}(E_G^0)$ is induced by the Yang–Mills–Higgs flow on $\mathcal{S}(W)$, and that the convergence of the flow on $\mathcal{S}(W)$ (guaranteed by the results of [19]) implies that the flow of YMH_G on $\mathcal{S}(E_G^0)$ converges also. The result then follows by showing that the Harder–Narasimhan–socle reduction on W induces the Harder–Narasimhan–socle reduction on E_G^0.

The paper is organized as follows. Section 2 contains the basic results necessary to the rest of the paper: the construction of the map $\phi : \mathcal{S}(E^0_G) \hookrightarrow \mathcal{S}(W)$ in equation (2.14), and a proof that the Yang–Mills–Higgs flows coincide (Corollary 2.4). The main result of Section 3 is Theorem 3.4, which shows that the flow of YMH_G on $\mathcal{S}(E^0_G)$ converges. The results of the final section relate the Harder–Narasimhan–socle reduction of a Higgs pair $(\overline{\partial}_{E^0_G}\,, \theta) \in \mathcal{S}(E^0_G)$ to that of the associated Higgs pair $\phi(\overline{\partial}_{E^0_G}\,, \theta) \in \mathcal{S}(W)$.

2. The Yang–Mills–Higgs functional

Let G be a connected reductive linear algebraic group defined over $\mathbb{C}$. Fix a maximal compact subgroup

$$K \subset G\,. \tag{2.1}$$

Fix a faithful representation

$$\rho\,:\, G \longrightarrow \mathrm{GL}(V)\,, \tag{2.2}$$

where V is a finite dimensional complex vector space. Fix a maximal compact subgroup $\widetilde{K}$

$$\rho(K) \subset \widetilde{K} \subset \mathrm{GL}(V) \tag{2.3}$$

of $\mathrm{GL}(V)$.

The Lie algebra of G will be denoted by $\mathfrak{g}$. The group G has the adjoint action on $\mathfrak{g}$. So $\mathfrak{g}$ is a G–module.

Let X be a compact connected Riemann surface. Fix a C^∞ principal K–bundle

$$E^0_K \longrightarrow X\,. \tag{2.4}$$

Let

$$E^0_G \,:=\, E^0_K(G) \,=\, E^0_K \times_K G \longrightarrow X \tag{2.5}$$

be the principal G–bundle obtained by extending the structure group of E^0_K using the inclusion map $K \hookrightarrow G$. Let $\mathrm{ad}(E^0_G) := E^0_G(\mathfrak{g}) = E^0_G \times_G \mathfrak{g}$ be the adjoint bundle of E^0_G. In other words, $\mathrm{ad}(E^0_G)$ is the vector bundle over X associated to the principal G–bundle E^0_G for the G–module $\mathfrak{g}$.

Let

$$\mathcal{A}_0 \,:=\, \mathcal{A}_{E^0_G} \tag{2.6}$$

be the space of all holomorphic structures on the principal G–bundle E^0_G. We note that $\mathcal{A}_0$ is an affine space for the vector space $A^{0,1}(\mathrm{ad}(E^0_G))$, which is the space of all smooth $(0\,,1)$–forms with values in $\mathrm{ad}(E^0_G)$. Fix a holomorphic structure

$$\overline{\partial}_0 \,:=\, \overline{\partial}^0_{E^0_G} \tag{2.7}$$

on E^0_G. Using $\overline{\partial}_0$, the affine space $\mathcal{A}_0$ gets identified with $A^{0,1}(\mathrm{ad}(E^0_G))$.

Let $E_G \longrightarrow X$ be a holomorphic principal G–bundle. A *Higgs field* on E_G is a holomorphic section of $\mathrm{ad}(E_G) \otimes K_X$ over X. A pair $(E_G\,, \theta)$, where θ is a Higgs field on E_G, is called a *Higgs G–bundle*. A holomorphic structure on a principal G–bundle E_G defines a holomorphic structure on the vector bundle $\mathrm{ad}(E_G) \otimes K_X$. The Dolbeault operator on $\mathrm{ad}(E^0_G) \otimes K_X$ corresponding to any $\overline{\partial}_{E^0_G} \in \mathcal{A}_0$ (see (2.6)) will also be denoted by $\overline{\partial}_{E^0_G}$.

We note that a pair $(\overline{\partial}_{E^0_G}\, , \theta) \in \mathcal{A}_0 \times A^{1,0}(\text{ad}(E^0_G))$ with $\overline{\partial}_{E^0_G}(\theta) = 0$ defines a Higgs G–bundle.

Define

$$\mathcal{B}_0 := \mathcal{A}_0 \times A^{1,0}(\text{ad}(E^0_G)) . \tag{2.8}$$

So if $\overline{\partial}_{E^0_G} \in \mathcal{A}_0$, and θ is a Higgs field on the holomorphic principal G–bundle $(E^0_G\, , \overline{\partial}_{E^0_G})$, then $(\overline{\partial}_{E^0_G}\, , \theta) \in \mathcal{B}_0$. We will see later that $\mathcal{B}_0$ is the total space of the cotangent bundle of the affine space $\mathcal{A}_0$.

Let

$$W := E^0_K(V) \longrightarrow X \tag{2.9}$$

be the vector bundle associated to the principal K–bundle E^0_K (see (2.4)) for the K–module V in (2.2). Therefore, W is identified with the vector bundle associated to the principal G–bundle E^0_G in (2.5) for the G–module V. A holomorphic structure on E^0_G defines a holomorphic structure on the vector bundle W. Using the injective homomorphism of Lie algebras associated to ρ in (2.2)

$$\mathfrak{g} \longrightarrow \text{End}_{\mathbb{C}}(V) , \tag{2.10}$$

we get a homomorphism of vector bundles

$$\text{ad}(E^0_G) \longrightarrow End(W) = W \otimes W^* , \tag{2.11}$$

where W is the vector bundle in (2.9). Take any $(\overline{\partial}_{E^0_G}\, , \theta) \in \mathcal{B}_0$ (see (2.8)). Let $\overline{\partial}_W$ be the holomorphic structure on W defined by $\overline{\partial}_{E^0_G}$. Let $\theta_W \in A^{1,0}(End(W))$ be the smooth section given by θ using the homomorphism in (2.11).

Let $\mathcal{A}(W)$ be the space of all holomorphic structures on the vector bundle W. Define

$$\mathcal{B}_W := \mathcal{A}(W) \times A^{1,0}(End(W)) . \tag{2.12}$$

Since W is associated to E^0_G by a faithful representation, there is a natural embedding

$$\delta : \mathcal{A}_0 \longrightarrow \mathcal{A}(W) , \tag{2.13}$$

where $\mathcal{A}_0$ is defined in (2.6). We have an embedding

$$\phi : \mathcal{B}_0 \longrightarrow \mathcal{B}_W \tag{2.14}$$

that sends any $(\overline{\partial}_{E^0_G}\, , \theta)$ to the pair $(\overline{\partial}_W\, , \theta_W)$ constructed above from $(\overline{\partial}_{E^0_G}\, , \theta)$.

The Lie algebra of $\widetilde{K}$ (see (2.3)) will be denoted by $\widetilde{\mathfrak{k}}$. Let g_0 denote the inner product on $\widetilde{\mathfrak{k}}$ defined by $\langle A\, , B\rangle = -\text{trace}(AB)$. Since $\text{End}_{\mathbb{C}}(V) = \widetilde{\mathfrak{k}} \oplus \sqrt{-1}\widetilde{\mathfrak{k}}$, where V is the G–module in (2.2), this g_0 defines a Hermitian inner product g_1 on $\text{End}_{\mathbb{C}}(V)$. The Lie algebra of K (see (2.1)) will be denoted by $\mathfrak{k}$. Let g'_0 be the restriction of g_0 to the subspace $\mathfrak{k} \subset \widetilde{\mathfrak{k}}$. Since $\mathfrak{g} = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{k}$, this g'_0 defines a Hermitian inner product g'_1 on $\mathfrak{g}$. Note that g'_1 is the restriction of g_1. The inner products g'_1 and g_1 induce inner products on the fibers of the vector bundle $\text{ad}(E^0_G)$ and $End(W)$ respectively. Indeed, these follow from the fact that g'_1 and g_1 are K–invariant and $\widetilde{K}$–invariant respectively.

We will now show that the Cartesian product $\mathcal{B}_0$ in (2.8) is the total space of the cotangent bundle of the affine space $\mathcal{A}_0$. For any $(\omega_{0,1}\, , \omega_{1,0}) \in A^{0,1}(\text{ad}(E^0_G)) \times A^{1,0}(\text{ad}(E^0_G))$, we have

$$\langle \omega_{0,1}\, , \omega_{1,0}\rangle \in A^{1,1}$$

using the inner product on the fibers of $\text{ad}(E_G^0)$. Consider the pairing

$$A^{0,1}(\text{ad}(E_G^0)) \times A^{1,0}(\text{ad}(E_G^0)) \longrightarrow \mathbb{C}$$

defined by

$$(\omega_{0,1}\,,\omega_{1,0}) \longmapsto \int_X \langle \omega_{0,1}\,,\omega_{1,0}\rangle \,\in\, \mathbb{C}\,.$$

This pairing identifies $A^{1,0}(\text{ad}(E_G^0))$ with the dual of $A^{0,1}(\text{ad}(E_G^0))$. Therefore, the total space of the cotangent bundle of the affine space $\mathcal{A}_0$ gets identified with $\mathcal{B}_0$.

Similarly, using the inner product on the fibers of $End(W)$, the Cartesian product $\mathcal{B}_W$ in (2.12) gets identified with the total space of the cotangent bundle of the affine space $\mathcal{A}(W)$.

Since the fibers of the vector bundles $\text{ad}(E_G^0)$ and $End(W)$ have inner products, we get inner products on the vector spaces

$$A^{0,1}(\text{ad}(E_G^0)) \oplus A^{1,0}(\text{ad}(E_G^0)) \quad \text{and} \quad A^{0,1}(End(W)) \oplus A^{1,0}(End(W))\,.$$

More precisely, the inner product on $A^{0,1}(\text{ad}(E_G^0)) \oplus A^{1,0}(\text{ad}(E_G^0))$ is defined by

$$\|(\omega_{0,1}\,,\omega_{1,0})\|^2 \,=\, \sqrt{-1}\int_X \langle \omega_{1,0}\,,\overline{\omega_{1,0}}\rangle - \sqrt{-1}\int_X \langle \omega_{0,1}\,,\overline{\omega_{0,1}}\rangle\,.$$

The inner product on $A^{0,1}(End(W)) \oplus A^{1,0}(End(W))$ is defined similarly.

Recall that $\mathcal{B}_0$ and $\mathcal{B}_W$ are identified with

$$A^{0,1}(\text{ad}(E_G^0)) \oplus A^{1,0}(\text{ad}(E_G^0)) \quad \text{and} \quad A^{0,1}(End(W)) \oplus A^{1,0}(End(W))$$

respectively (the affine space $\mathcal{A}_0$ is identified with $A^{0,1}(\text{ad}(E_G^0))$ after choosing the base point $\overline{\partial}_0$ in (2.7); since $\overline{\partial}_0$ gives a point in $\mathcal{A}(W)$, it follows that $\mathcal{A}(W)$ is identified with $A^{0,1}(End(W))$). Therefore, the inner products on $A^{0,1}(\text{ad}(E_G^0)) \oplus A^{1,0}(\text{ad}(E_G^0))$ and $A^{0,1}(End(W)) \oplus A^{1,0}(End(W))$ define Kähler structures on $\mathcal{B}_0$ and $\mathcal{B}_W$ respectively.

Lemma 2.1. *The embedding ϕ in* (2.14) *preserves the Kähler forms. Moreover, the second fundamental form of the embedding ϕ vanishes. In particular, this embedding is totally geodesic.*

Proof. Since the inner product g_1' on $\mathfrak{g}$ is the restriction of the inner product g_1 on $\text{End}_{\mathbb{C}}(V)$, it follows immediately that ϕ preserves the Kähler forms.

Let $\mathfrak{g}^\perp \subset \text{End}_{\mathbb{C}}(V)$ be the orthogonal complement for the inner product g_1 on $\text{End}_{\mathbb{C}}(V)$. Since g_1 is K–invariant (recall that it is in fact $\widetilde{K}$–invariant), and the adjoint action of K on $\text{End}_{\mathbb{C}}(V)$ preserves the subspace $\mathfrak{g} \subset \text{End}_{\mathbb{C}}(V)$, it follows that the adjoint action of K on $\text{End}_{\mathbb{C}}(V)$ preserves $\mathfrak{g}^\perp$. Since K is Zariski dense in G, it follows that the adjoint action of G on $\text{End}_{\mathbb{C}}(V)$ preserves $\mathfrak{g}^\perp$. Therefore, the orthogonal decomposition

$$\text{End}_{\mathbb{C}}(V) \,=\, \mathfrak{g} \oplus \mathfrak{g}^\perp \tag{2.15}$$

is preserved by the adjoint action of G.

Let

$$F_0 \,:=\, E_K^0(\mathfrak{g}^\perp) \longrightarrow X \tag{2.16}$$

be the vector bundle associated to the principal K–bundle E^0_K (see (2.4)) for the K–module $\mathfrak{g}^\perp$. Note that the G–invariant orthogonal decomposition of $\mathrm{End}_{\mathbb{C}}(V)$ in (2.15) induces an orthogonal decomposition

$$End(W) = \mathrm{ad}(E^0_G) \oplus F_0 \, . \tag{2.17}$$

Hence we have orthogonal decompositions

$$A^{0,1}(End(W)) = A^{0,1}(\mathrm{ad}(E^0_G)) \oplus A^{0,1}(F_0) \text{ and } A^{1,0}(End(W)) = A^{1,0}(\mathrm{ad}(E^0_G)) \oplus A^{1,0}(F_0) \, . \tag{2.18}$$

Let $\mathcal{H}$ denote the trivial vector bundle over $\mathcal{B}_0$ (see (2.8)) with fiber $A^{0,1}(F_0) \oplus A^{1,0}(F_0)$. Using the orthogonal decompositions in (2.18) it follows that the orthogonal complement of the differential

$$d\phi \,:\, T^{1,0}\mathcal{B}_0 \longrightarrow \phi^* T^{1,0}\mathcal{B}_W \tag{2.19}$$

is identified with the above defined vector bundle $\mathcal{H}$ (here $T^{1,0}$ denotes the holomorphic tangent bundle).

On the other hand, $\mathcal{H} \subset \phi^* T^{1,0}\mathcal{B}_W$ is a holomorphic subbundle because the adjoint action of G on $\mathrm{End}_{\mathbb{C}}(V)$ preserves $\mathfrak{g}^\perp$. Consequently, the orthogonal complement $\mathcal{H} = d\phi(T^{1,0}\mathcal{B}_0)^\perp \subset \phi^* T^{1,0}\mathcal{B}_W$ (see (2.19)) is preserved by the Chern connection on the holomorphic Hermitian vector bundle $\phi^* T^{1,0}\mathcal{B}_W$. Since the Chern connection for a Kähler metric coincides with the Levi–Civita connection, it follows that $d\phi(T^{1,0}\mathcal{B}_0)^\perp$ is preserved by the connection on $\phi^* T^{1,0}\mathcal{B}_W$ obtained by pulling back the Levi–Civita connection on $\mathcal{B}_W$. In other words, the second fundamental form of the embedding ϕ vanishes. This completes the proof of the lemma. $\square$

Remark 2.2. Take any $z := (\overline{\partial}_{E^0_G}, \theta) \in \mathcal{B}_0$ which is a Higgs G–bundle, meaning θ is holomorphic with respect to the holomorphic structure $\overline{\partial}_{E^0_G}$. The $\phi(z)$ is a Higgs vector bundle.

A connection on the principal G–bundle E^0_G decomposes the real tangent bundle $T^{\mathbb{R}}E^0_G$ into a direct sum of horizontal and vertical tangent bundles. Using this decomposition, the almost complex structures of G and X together produce an almost complex structure on E^0_G. Let $\overline{\partial}_{E^0_G}$ be a holomorphic structure on E^0_G. A connection ∇ on E^0_G is said to be *compatible* with $\overline{\partial}_{E^0_G}$ if the almost complex structure on E^0_G given by ∇ coincides with the one underlying the complex structure $\overline{\partial}_{E^0_G}$ on E^0_G.

Given a holomorphic structure $\overline{\partial}_{E^0_G}$ on E^0_G, there is a unique connection ∇ on E^0_K such that the connection on E^0_G induced by ∇ is compatible with $\overline{\partial}_{E^0_G}$; it is known as the *Chern connection*. On the other hand, given a connection ∇_1 on E^0_G, there is a unique holomorphic structure $\overline{\partial}'_1$ on E^0_G such that ∇_1 is compatible with $\overline{\partial}'_1$ (this is because $\dim_{\mathbb{C}} X = 1$). Therefore, we have a canonical bijective correspondence between $\mathcal{A}_0$ (see (2.6)) and the space of all connections on E^0_K. Similarly, we have a canonical bijective correspondence between $\mathcal{A}(W)$ (the space of all holomorphic structures on the vector bundle W in (2.9)) and the space of all connections on the principal $\widetilde{K}$–bundle

$$E^0_{\widetilde{K}} := E^0_K(\widetilde{K}) = E^0_K \times_K \widetilde{K} \longrightarrow X \tag{2.20}$$

obtained by extending the structure group of E^0_K using the homomorphism $\rho : K \longrightarrow \widetilde{K}$.

The curvature of a connection ∇ will be denoted by $K(\nabla)$.

Let

$$* \,:\, End(W) \,\longrightarrow\, End(W) \tag{2.21}$$

be the conjugate linear automorphism that acts on the subbundle $\mathrm{ad}(E^0_{\widetilde{K}}) \,\subset\, End(W)$ (see (2.20)) as multiplication by -1; since $End(W) \,=\, \mathrm{ad}(E^0_{\widetilde{K}}) \oplus \sqrt{-1}\cdot \mathrm{ad}(E^0_{\widetilde{K}})$, this condition uniquely determines the automorphism in (2.21).

Fix a Hermitian metric h_0 on $T^{1,0}X$.

Let

$$\mathrm{YMH}_W \,:\, \mathcal{B}_W \,\longrightarrow\, \mathbb{R}$$

be the function defined by $(\overline{\partial}_{E^0_G}\,,\theta) \,\longmapsto\, \|K(\overline{\partial}_{E^0_G}) + [\theta\,,\theta^*]\|^2$, where $K(\overline{\partial}_{E^0_G})$ is the curvature of the connection associated to $\overline{\partial}_{E^0_G}$, and the inner product on 2–forms is defined using h and the inner product on the fibers of $End(E)$. If locally $\theta \,=\, A\times dz$, then $[\theta\,,\theta^*] \,=\, (AA^* - A^*A)dz{\wedge}d\overline{z}$; see [19] for more on this function YMH_W.

Let $* \,:\, \mathrm{ad}(E^0_G) \,\longrightarrow\, \mathrm{ad}(E^0_G)$ be the conjugate linear automorphism that acts on the subbundle $\mathrm{ad}(E^0_K) \,\subset\, \mathrm{ad}(E^0_G)$ (see (2.4)) as multiplication by -1. Note that this automorphism coincides with the restriction of the automorphism in (2.21). Consider $\mathcal{B}_0$ defined in (2.8). Let

$$\mathrm{YMH}_G \,:\, \mathcal{B}_0 \,\longrightarrow\, \mathbb{R} \tag{2.22}$$

be the function defined by $(\overline{\partial}_{E^0_G}\,,\theta) \,\longmapsto\, \|K(\overline{\partial}_{E^0_G}) + [\theta\,,\theta^*]\|^2$; as before, $K(\overline{\partial}_{E^0_G})$ is the curvature of the connection associated to the holomorphic structure $\overline{\partial}_{E^0_G}$.

We first note that

$$\mathrm{YMH}_G \,=\, \mathrm{YMH}_W \circ \phi\,, \tag{2.23}$$

where ϕ is the function constructed in (2.14). Let $d\mathrm{YMH}_W$ be the smooth exact 1–form on $\mathcal{B}_W$. The following lemma shows that the normal vectors to $T^{\mathbb{R}}\mathcal{B}_0$ for the embedding ϕ in (2.14) are annihilated by the form $d\mathrm{YMH}_W$.

Lemma 2.3. *For any point $x \,\in\, \mathcal{B}_0$, and any normal vector*

$$v \,\in\, (d\phi(T^{\mathbb{R}}_x\mathcal{B}_0))^{\perp} \,\subset\, T^{\mathbb{R}}_{\phi(x)}\mathcal{B}_W\,,$$

the following holds:

$$d\mathrm{YMH}_W(v) \,=\, 0\,.$$

Proof. Take any pair $(v\,,w) \,\in\, A^{0,1}(F_0)\oplus A^{1,0}(F_0)$, where F_0 is the vector bundle in (2.16). Take any $(\overline{\partial}_{E^0_G}\,,\theta) \,\in\, \mathcal{B}_0$. Let ∇ be the connection on E^0_G corresponding to the holomorphic structure $\overline{\partial}_{E^0_G}$. Therefore, the connection on E^0_G corresponding to the holomorphic structure $\overline{\partial}_{E^0_G} + tv$, where $t \,\in\, \mathbb{R}$, is $\nabla + tv - tv^*$. The automorphism in (2.21) preserves the orthogonal decomposition of $End(W)$ in (2.17). Hence for $t \,\in\, \mathbb{R}$, all four tv, tv^*, tw and tw^* are 1–forms with values in F_0. On the other hand, θ and θ^* are 1–forms with values in $\mathrm{ad}(E^0_G)$.

Let ∇' be the connection on the vector bundle W associated to E^0_G induced by the connection ∇ on E^0_G. So the curvature of ∇' coincides with the curvature of ∇, in particular, $K(\nabla')$ is a 2–form with values in $\mathrm{ad}(E^0_G)$. We note that

$$K(\nabla' + tv - tv^*) \,=\, K(\nabla') + t\nabla'(v - v^*) + t^2 C\,,$$

where C is independent of t. Since ∇' is induced by a connection E^0_G, and the decomposition in (2.15) is preserved by the action of G, the connection ∇' preserves the decomposition in (2.17). Hence $\nabla'(v - v^*)$ is a 2–form with values in F_0.

Using these and the fact that the decompositions in (2.18) are orthogonal, we have

$$(\frac{d}{dt}\|K(\nabla' + tv - tv^*) + [\theta + tw\,, \theta^* + tw^*]\|^2)\Big|_{t=0} \,=\, 0\,.$$

This completes the proof of the lemma. □

Let

$$\Psi_W \,:\, \mathcal{B}_W \,\longrightarrow\, T^{\mathbb{R}}\mathcal{B}_W \tag{2.24}$$

be the gradient vector field on $\mathcal{B}_W$ for the function YMH_W. From Lemma 2.3 and (2.23) we have the following corollary:

Corollary 2.4. *The restriction of the vector field Ψ_W to $\phi(\mathcal{B}_0)$ (see (2.14)) lies in the image of the differential $d\phi$ in (2.19). Furthermore, this restriction coincides with the gradient vector field for the function YMH_G.*

3. Closedness of the embedding

For a complex vector space V', let $P(V')$ denote the projective space of lines in V'. Any linear action on V' induces an action on $P(V')$.

Consider the closed subgroup $\rho(G) \subset \mathrm{GL}(V)$ in (2.2). A theorem of C. Chevalley (see [12, p. 80]) says that there is a finite dimensional left $\mathrm{GL}(V)$–module V_1 and a line

$$\ell \subset V_1 \tag{3.1}$$

such that $\rho(G)$ is exactly the isotropy subgroup, for the action of $\mathrm{GL}(V)$ on $P(V_1)$, of the point in $P(V_1)$ representing the line ℓ.

Let $E_{\mathrm{GL}(V)} := E^0_G(\mathrm{GL}(V)) = E^0_G \times_G \mathrm{GL}(V) \longrightarrow X$ be the principal $\mathrm{GL}(V)$–bundle obtained by extending the structure group of E^0_G (see (2.5)) by the homomorphism ρ in (2.2). Therefore, the vector bundle $E_{\mathrm{GL}(V)}(V)$, associated to $E_{\mathrm{GL}(V)}$ by the standard action of $\mathrm{GL}(V)$ on V, is identified with the vector bundle W in (2.9). Let

$$\mathcal{V}_1 \,:=\, E_{\mathrm{GL}(V)}(V_1) \,\longrightarrow\, X \tag{3.2}$$

be the vector bundle associated to $E_{\mathrm{GL}(V)}$ for the above $\mathrm{GL}(V)$–module V_1. Since

$$\mathcal{V}_1 \,=\, E^0_G(V_1)\,,$$

and the action of G on V_1 preserves the line ℓ in (3.1), the line ℓ defines a C^∞ line subbundle

$$L_0 \subset \mathcal{V}_1\,. \tag{3.3}$$

Take any holomorphic structure $\overline{\partial}_W \in \mathcal{A}(W)$ on the vector bundle W (see (2.13)). The holomorphic structure $\overline{\partial}_W$ on W defines a holomorphic structure on the principal $\mathrm{GL}(V)$–bundle $E_{\mathrm{GL}(V)}$ corresponding to W. Hence $\overline{\partial}_W$ defines a holomorphic structure on the associated vector bundle $\mathcal{V}_1$ in (3.2). This holomorphic structure on $\mathcal{V}_1$ will be denoted by $\overline{\partial}'_1$.

Since $\rho(G)$ is the isotropy subgroup of the point in $P(V_1)$ representing the line ℓ for the action of $\mathrm{GL}(V)$ on $P(V_1)$, we conclude that $\overline{\partial}_W$ lies in $\delta(\mathcal{A}_0)$ (see (2.13)) if and only if the line subbundle L_0 in (3.3) is holomorphic with respect to the holomorphic structure $\overline{\partial}'_1$ on $\mathcal{V}_1$.

Therefore, we have the following lemma:

Lemma 3.1. *The embedding δ in* (2.13) *is closed.*

The action of $\mathrm{GL}(V)$ on V_1 gives a homomorphism

$$\mathrm{End}_{\mathbb{C}}(V) \longrightarrow \mathrm{End}_{\mathbb{C}}(V_1)$$

of Lie algebras. This homomorphism in turn gives a homomorphism of vector bundles

$$End(W) \longrightarrow End(\mathcal{V}_1)\,, \tag{3.4}$$

where $\mathcal{V}_1$ is the vector bundle in (3.2).

Take any $\theta \in A^{1,0}(End(W))$. Let $\theta' \in A^{1,0}(End(\mathcal{V}_1))$ be the section constructed from θ using the homomorphism in (3.4). Since $\rho(G)$ is the isotropy subgroup of the point in $P(V_1)$ representing the line ℓ for the action of $\mathrm{GL}(V)$ on $P(V_1)$, we conclude the following: The section θ lies in the image of the natural homomorphism

$$A^{1,0}(\mathrm{ad}(E^0_G)) \longrightarrow A^{1,0}(End(W))$$

if and only if $\theta'(L_0) \in A^{1,0}(L_0)$, where L_0 is the subbundle in (3.3).

Therefore, using Lemma 3.1, we have following proposition:

Proposition 3.2. *The embedding ϕ in* (2.14) *is closed.*

Let

$$\mathcal{S}(E^0_G) \subset \mathcal{B}_0 \tag{3.5}$$

be the subset consisting of all pairs that are Higgs G–bundles. So a pair

$$(\overline{\partial}_{E^0_G}\,, \theta) \in \mathcal{A}_0 \times A^{1,0}(\mathrm{ad}(E^0_G)) = \mathcal{B}_0$$

lies in $\mathcal{S}(E^0_G)$ if and only if the section θ is holomorphic with respect to the holomorphic structure on $\mathrm{ad}(E^0_G) \otimes K_X$ defined by $\overline{\partial}_{E^0_G}$.

Consider the gradient flow on $\mathcal{B}_0$ for the function YMH_G defined in (2.22). The following lemma shows that this flow preserves the subset $\mathcal{S}(E^0_G)$ defined in (3.5).

Lemma 3.3. *Take any $z := (\overline{\partial}_{E^0_G}\,, \theta) \in \mathcal{S}(E^0_G)$. Let*

$$\gamma_z : \mathbb{R} \longrightarrow \mathcal{B}_0$$

be the integral curve for the gradient flow on $\mathcal{B}_0$ for the function YMH_G *such that $\gamma_z(0) = z$. Then*

$$\gamma_z(t) \in \mathcal{S}(E^0_G)$$

for all $t \in \mathbb{R}$.

Proof. Consider $\mathcal{B}_W$ defined in (2.12). Let

$$\mathcal{S}(W) \subset \mathcal{B}_W$$

be the subset consisting of all pairs $(\overline{\partial}', \theta) \in \mathcal{A}(W) \times A^{1,0}(End(W))$ such that θ is holomorphic with respect to the holomorphic structure given by $\overline{\partial}'$. For the map ϕ in (2.14),

$$\phi(\mathcal{S}(E_G^0)) \subset \mathcal{S}(W)$$

(see Remark 2.2).

In view of Corollary 2.4, to prove the lemma it suffices to show that the vector field Ψ_W (defined in (2.24)) preserves the subset $\mathcal{S}(W)$. But this is proved in [19]; from [19, Lemma 3.10] and the proof of Proposition 3.2 in [19, pp. 295–297] it follows that the flow Ψ_W is generated by the action of the complex gauge group, hence $\mathcal{S}(W)$ is preserved by the flow. This completes the proof of the lemma. □

Theorem 3.4. *The integral curve γ_z for the gradient flow of* YMH_G *on $\mathcal{B}_0$ with initial condition $z := (\overline{\partial}_{E_G^0}, \theta) \in \mathcal{S}(E_G^0)$ converges to a limit in $\mathcal{S}(E_G^0)$.*

Proof. Theorem 1.1 in [19] shows that the gradient flow of YMH_W on the space $\mathcal{B}_W$ with initial conditions in $\mathcal{S}(W)$ converges to a limit in $\mathcal{S}(W)$. Moreover, Corollary 2.4 and Lemma 3.3 together with the uniqueness of the flow from Proposition 3.2 in [19] give the following: when the initial conditions are in $\phi(\mathcal{S}(E_G^0))$, then the flow preserves the space $\phi(\mathcal{S}(E_G^0))$. Combining these two facts, we see that because the embedding ϕ is closed by Proposition 3.2, the limit of the flow lies in $\phi(\mathcal{S}(E_G^0))$. Since $\phi(\gamma_z)$ coincides with the gradient flow of YMH_W with initial conditions in $\phi(\mathcal{S}(E_G^0))$ by Corollary 2.4, we conclude that $\lim_{t\to\infty} \gamma_z(t)$ exists, and it is in $\mathcal{S}(E_G^0)$. □

4. Reduction of structure group

As before, G is a connected reductive linear algebraic group defined over $\mathbb{C}$.

See [3], [6] for the definitions of semistable, stable and polystable Higgs G–bundles.

Lemma 4.1. *Let (E_G, θ) be a semistable Higgs G–bundle on X. The Higgs vector bundle $(\mathrm{ad}(E_G), \varphi)$ is semistable, where φ is the Higgs field on $\mathrm{ad}(E_G)$ defined by θ using the Lie algebra structure of the fibers of $\mathrm{ad}(E_G)$.*

Proof. This follows from [3, p. 37, Lemma 3.6], but some explanations are necessary.

Let $Z(G) \subset G$ be the connected component of the center of G containing the identity element. Define

$$G' := G/Z(G).$$

Let $(E_{G'}, \theta')$ be the Higgs G'–bundle over X obtained by extending the structure group of (E_G, θ) using the quotient map $G \longrightarrow G'$. Since (E_G, θ) is semistable, it follows immediately that the Higgs G'–bundle $(E_{G'}, \theta')$ is semistable. Let φ' be the Higgs field on the adjoint vector bundle $\mathrm{ad}(E_{G'})$ induced by θ'. Since $(E_{G'}, \theta')$ is semistable, and the group G' does not have any nontrivial character, the Higgs vector bundle $(\mathrm{ad}(E_{G'}), \varphi')$ is semistable [3, p. 37, Lemma 3.6] (see also [3, p. 26, Proposition 2.4]). We have

$$(\mathrm{ad}(E_G), \varphi) = (\mathrm{ad}(E_{G'}), \varphi') \oplus (X \times z(\mathfrak{g}), 0),$$

where $z(\mathfrak{g})$ is the Lie algebra of $Z(G)$, and $X \times z(\mathfrak{g})$ is the trivial vector bundle over X with fiber $z(\mathfrak{g})$. Hence $(\text{ad}(E_G)\,,\varphi)$ is semistable; note that $\text{degree}(\text{ad}(E_G)) = 0 = \text{degree}(\text{ad}(E_{G'}))$. This completes the proof of the lemma. $\square$

Let $(E_G\,,\theta)$ be a Higgs G–bundle on X and $H \subset G$ a closed algebraic subgroup. A *reduction* of structure group of the Higgs G–bundle $(E_G\,,\theta)$ to H is a holomorphic reduction of structure group $E_H \subset E_G$ to H over X such that θ lies in the image of the homomorphism $H^0(X,\, \text{ad}(E_H)\otimes K_X) \longrightarrow H^0(X,\, \text{ad}(E_G)\otimes K_X)$.

Given a Higgs G–bundle $(E_G\,,\theta)$, there is a canonical Harder–Narasimhan reduction of structure group of $(E_G\,,\theta)$ to a parabolic subgroup P of G [10] (the method in [10] is based on [4]). If $(E_G\,,\theta)$ is semistable, then $P = G$.

We recall the definition of the Harder–Narasimhan reduction of a Higgs G–bundle.

Let $(E_G\,,\theta)$ be a Higgs G–bundle on X. Then there is a parabolic subgroup $P \subset G$ and a reduction of structure group E_P of $(E_G\,,\theta)$ to P such that

(1) the principal $L(P)$–bundle $E_{L(P)} := E_P\times_P L(P) \longrightarrow X$, where $L(P)$ is the Levi quotient of P, is semistable, and
(2) for any nontrivial character χ of P which is a nonnegative linear combination of simple roots (with respect to some Borel subgroup contained in P) and is trivial on the center of G, the associated line bundle $E_P(\chi) \longrightarrow X$ is of positive degree.

The above pair $(P\,,E_P)$ is unique in the following sense: for any other pair $(P_1\,,E_{P_1})$ satisfying the above two conditions, there is some $g \in G$ such that

- $P_1 = g^{-1}Pg$, and
- $E_{P_1} = E_P g$.

(See [10], [4].)

A semistable vector bundle $E \longrightarrow X$ admits a filtration of subbundles

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n = E \tag{4.1}$$

such that E_i/E_{i-1}, $1 \leq i \leq n$, is the maximal polystable subbundle of E/E_{i-1} with

$$\frac{\text{degree}(E_i/E_{i-1})}{\text{rank}(E_i/E_{i-1})} = \frac{\text{degree}(E)}{\text{rank}(E)}$$

(see [13, p. 23, Lemma 1.5.5]); this filtration is called the *socle filtration*. In [1], this was generalized to semistable principal G–bundles (see [1, p. 218, Proposition 2.12]). In Theorem 4.4 proved below, this is further generalized to semistable Higgs G–bundles.

We will define admissible reductions of a Higgs G–bundle. See [5, pp. 3998–3999] for the definition of an admissible reduction of structure group of a principal G–bundle.

Definition 4.2. An *admissible* reduction of structure group of a Higgs G–bundle $(E_G\,,\theta)$ to a parabolic subgroup $P \subset G$ is a reduction of structure group E_P of $(E_G\,,\theta)$ to P such that $E_P \subset E_G$ is an admissible reduction of E_G.

Let $(E_G\,,\theta)$ be a Higgs G–bundle on X. Let $E'_P \subset E_G$ be a reduction of structure group of $(E_G\,,\theta)$ to a parabolic subgroup P of G. So θ is a section of $\text{ad}(E'_P)\otimes K_X$. Let $L(P)$ be the Levi quotient of P. Let $E'_P(L(P))$ be the principal $L(P)$–bundle over X obtained by extending the structure group of E'_P using the quotient map $P \longrightarrow L(P)$.

The quotient homomorphism $\text{Lie}(P) \longrightarrow \text{Lie}(L(P))$ induces a homomorphism of adjoint bundles

$$\text{ad}(E'_P) \longrightarrow \text{ad}(E'_P(L(P)))\,.$$

Using this homomorphism of vector bundles, the section θ of $\text{ad}(E'_P)\otimes K_X$ gives a holomorphic section of $\text{ad}(E'_P(L(P)))\otimes K_X$. In other words, θ gives a Higgs field on $E'_P(L(P))$. This Higgs field on $E'_P(L(P))$ will be denoted by θ'.

Let $(E_G\,,\theta)$ be a semistable Higgs G–bundle on X which is not polystable. Let $Q \subsetneq G$ be a proper parabolic subgroup which is maximal among all the proper parabolic subgroups P such that $(E_G\,,\theta)$ has an admissible reduction of structure group $E'_P \subset E_G$ (see Definition 4.2) for which the associated Higgs $L(P)$–bundle $(E'_P(L(P))\,,\theta')$ defined above is polystable.

Definition 4.3. An admissible reduction of structure group of $(E_G\,,\theta)$ to Q

$$E_Q \subset E_G$$

will be called a *socle reduction* if the associated Higgs $L(Q)$–bundle $(E_Q(L(Q))\,,\theta')$ is polystable, where $L(Q)$ is the Levi quotient of Q.

Theorem 4.4. *Let $(E_G\,,\theta)$ be a semistable Higgs G–bundle on X which is not polystable. Then $(E_G\,,\theta)$ admits a socle reduction. If $(Q\,,E_Q)$ and $(Q_1\,,E_{Q_1})$ are two socle reductions of $(E_G\,,\theta)$, then there is some $g \in G$ such that $Q_1 = g^{-1}Qg$, and $E_{Q_1} = E_Q g$.*

Proof. First note that the construction of the socle filtration of a semistable vector bundle extends to semistable Higgs bundles; indeed, the proof in [13, p. 23, Lemma 1.5.5] goes through in this case also. Therefore, if $(E\,,\theta)$ is a semistable Higgs vector bundle on X which is not polystable, there is filtration of subbundles

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n = E \tag{4.2}$$

such that

$$\theta(E_i) \subset E_i \otimes K_X \tag{4.3}$$

for all $i \in [1\,,n]$, and $(E_i/E_{i-1}\,,\theta'_i)$ is the unique maximal polystable Higgs subbundle of the Higgs bundle $(E/E_{i-1}\,,\theta''_i)$ such that

$$\frac{\text{degree}(E_i/E_{i-1})}{\text{rank}(E_i/E_{i-1})} = \frac{\text{degree}(E)}{\text{rank}(E)}\,,$$

where θ'_i and θ''_i are the Higgs fields on E_i/E_{i-1} and E/E_{i-1} respectively induced by θ (the condition in (4.3) ensures that θ induces Higgs fields on both E_i/E_{i-1} and E/E_{i-1}).

Let $\text{ad}(E_G) \longrightarrow X$ be the adjoint bundle of E_G. Let φ be the Higgs field on $\text{ad}(E_G)$ defined by θ. From Lemma 4.1 we know that the Higgs vector bundle $(\text{ad}(E_G)\,,\varphi)$ is semistable. We note if $(\text{ad}(E_G)\,,\varphi)$ is polystable, then $(E_G\,,\theta)$ is polystable. Since $(E_G\,,\theta)$ is not polystable, we conclude that the Higgs vector bundle $(\text{ad}(E_G)\,,\varphi)$ is not polystable. Let

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{m-1} \subset E_m = \text{ad}(E_G) \tag{4.4}$$

be the socle filtration for $(\text{ad}(E_G)\,,\varphi)$ (see (4.2)).

Fix a G–invariant nondegenerate symmetric bilinear form B_0 on the Lie algebra $\mathfrak{g}$ of G; such a form exists because G is reductive. This form B_0 defines nondegenerate symmetric bilinear forms on the fibers of $\text{ad}(E_G)$. So we get an isomorphism

$$\text{ad}(E_G) \stackrel{\sim}{\longrightarrow} \text{ad}(E_G)^* . \tag{4.5}$$

Let φ^* be the dual Higgs field on $\text{ad}(E_G)^*$ defined by φ. The isomorphism in (4.5) clearly takes φ to φ^*. In particular, $(\text{ad}(E_G)\, , \varphi)$ is self–dual.

From the uniqueness of the socle filtration it follows that the filtration in (4.4) is self–dual. Also, the integer m in (4.4) is odd. The tensor product of two semistable Higgs bundles on X is again semistable [18, p. 38, Corollary 3.8]. Using these observations it follows that

- the subbundle $E_{\frac{m+1}{2}}$ is closed under the Lie bracket operation on the fibers of $\text{ad}(E_G)$,
- the fibers of $E_{\frac{m-1}{2}}$ are ideals in the fibers of $E_{\frac{m+1}{2}}$, and are nilpotent,
- the fibers of the quotient $E_{\frac{m+1}{2}}/E_{\frac{m-1}{2}}$ are reductive, and
- the Higgs field θ is a section of $E_{\frac{m+1}{2}} \otimes K_X$.

(see [1, p. 218, Proposition 2.12]). It should be clarified that to prove the above statements we need the following: for any two polystable Higgs vector bundles $(W_1\, , \varphi_1)$ and $(W_2\, , \varphi_2)$ over X, the tensor product $(W_1 \otimes W_2\, , \varphi_1 \otimes \text{Id}_{W_2} + \text{Id}_{W_1} \otimes \varphi_2)$ is also a polystable Higgs vector bundle. To prove that $(W_1 \otimes W_2\, , \varphi_1 \otimes \text{Id}_{W_2} + \text{Id}_{W_1} \otimes \varphi_2)$ is polystable, let ∇_1 and ∇_2 be the Hermitian–Yang–Mills connections on $(W_1\, , \varphi_1)$ and $(W_2\, , \varphi_2)$ respectively (see [18, p. 19, Theorem 1(2)]). Then the induced connection $\nabla_1 \otimes \text{Id}_{W_2} + \text{Id}_{W_1} \otimes \nabla_2$ on $W_1 \otimes W_2$ is a Hermitian–Yang–Mills connection for $(W_1 \otimes W_2\, , \varphi_1 \otimes \text{Id}_{W_2} + \text{Id}_{W_1} \otimes \varphi_2)$. Hence $(W_1 \otimes W_2\, , \varphi_1 \otimes \text{Id}_{W_2} + \text{Id}_{W_1} \otimes \varphi_2)$ is polystable [18, p. 19, Theorem 1(2)].

From the above statements it follows that $E_{\frac{m+1}{2}}$ is a Lie algebra subbundle of the Lie algebra bundle $\text{ad}(E_G)$ such that the fibers of $E_{\frac{m+1}{2}}$ are parabolic subalgebras.

The fibers of $\text{ad}(E_G)$ are identified with the Lie algebra $\mathfrak{g}$ up to an inner automorphism. More precisely, for any point $x \in X$, and any point z in the fiber $(E_G)_x$ of E_G, we have an isomorphism

$$\sigma_z \,:\, \mathfrak{g} \longrightarrow \text{ad}(E_G)_x \tag{4.6}$$

that sends any $v \in \mathfrak{g}$ to the image of $(z\, , v)$ in $\text{ad}(E_G)_x$ (recall that $\text{ad}(E_G)$ is a quotient of $E_G \times \mathfrak{g}$). For any $g \in G$, the isomorphisms σ_z and σ_{zg} differ by the inner automorphism $\text{Ad}(g)$ of $\mathfrak{g}$. Let $Q \subset G$ be a parabolic subgroup in the conjugacy class of parabolic subgroups whose Lie algebras are identified with the fibers of $E_{\frac{m+1}{2}}$ by some isomorphism constructed in (4.6). The normalizer of any parabolic subgroup $P \subset G$ coincides with P. In particular, the normalizer of $Q \subset G$ is Q itself. Hence the subalgebra bundle $E_{\frac{m+1}{2}} \subset \text{ad}(E_G)$ gives a holomorphic reduction of structure group $E_Q \subset E_G$ such that the subbundle $\text{ad}(E_Q) \subset \text{ad}(E_G)$ coincides with $E_{\frac{m+1}{2}}$. For any point $x \in X$, the fiber $(E_Q)_x \subset (E_G)_x$ consists of all points $z \in (E_G)_x$ such that the isomorphism σ_z in (4.6) takes $\text{Lie}(Q)$ to $(E_{\frac{m+1}{2}})_x$.

Since $\text{ad}(E_Q) = E_{\frac{m+1}{2}}$, and the Higgs field θ is a section of $E_{\frac{m+1}{2}} \otimes K_X$, we conclude that E_Q is a reduction of structure group of the Higgs G–bundle (E_G, θ). It is straight–forward to check that E_Q is a socle reduction of (E_G, θ).

Given any socle reduction $E_{Q'}$ of (E_G, θ), it can be shown that the adjoint bundle $\text{ad}(E_{Q'})$ coincides with the subbundle $E_{\frac{m+1}{2}}$ in (4.4). From this the uniqueness statement in the theorem follows. This completes the proof of the theorem. □

For a polystable Higgs G–bundle (E_G, θ) the *socle reduction* is defined to be E_G itself.

Given a Higgs G–bundle, combining the Harder–Narasimhan reduction with the socle reduction we get a new Higgs G–bundle which will be described below.

Let (E_G, θ) be a Higgs G–bundle. Let (E_P, θ_P) be the Harder–Narasimhan reduction of (E_G, θ). If (E_G, θ) is semistable, then $P = G$, and $(E_P, \theta_P) = (E_G, \theta)$.

Let $L(P)$ be the Levi quotient of P. Let

$$(E_{L(P)}, \theta_{L(P)}) \tag{4.7}$$

be the Higgs $L(P)$–bundle obtained by extending the structure group of the above Higgs P–bundle (E_P, θ_P) using the quotient map $P \longrightarrow L(P)$. From the definition of a Harder–Narasimhan reduction we know that the Higgs $L(P)$–bundle $(E_{L(P)}, \theta_{L(P)})$ is semistable. Therefore, $(E_{L(P)}, \theta_{L(P)})$ has a unique socle reduction by Theorem 4.4. Let

$$E_H \subset E_{L(P)}$$

be the socle reduction of $(E_{L(P)}, \theta_{L(P)})$. So H is a Levi subgroup of a parabolic subgroup of $L(P)$; the Higgs field on E_H induced by $\theta_{L(P)}$ will be denoted by θ_H (see Definition 4.3).

The Levi quotient $L(P)$ is identified with all the Levi factors of P, and H is a subgroup of $L(P)$. Therefore, H becomes a subgroup of G after fixing a Levi factor of P. Let

$$(E'_G, \theta') \tag{4.8}$$

be the Higgs G–bundle obtained by extending the structure group of the Higgs H–bundle (E_H, θ_H) using the inclusion of H in G.

Take any

$$z := (\overline{\partial}_{E_G^0}, \theta) \in \mathcal{S}(E_G^0) \tag{4.9}$$

(see (3.5)). Let (E_G, θ) be the Higgs G–bundle defined by the holomorphic structure $\overline{\partial}_{E_G^0}$ on E_G^0 together with the section θ in (4.9). Let (E'_G, θ') be the new Higgs G–bundle constructed in (4.8) from (E_G, θ).

Lemma 4.5. *Let γ_z be the integral curve for the gradient flow of YMH_G on $\mathcal{B}_0$ with initial condition z (see (4.9)). Let*

$$(\overline{\partial}_1, \theta_1) = \lim_{t\to\infty} \gamma_z(t) \in \mathcal{S}(E_G^0)$$

be the limit in Theorem 3.4. Then the Higgs G–bundle defined by $(\overline{\partial}_1, \theta_1)$ is holomorphically isomorphic to the Higgs G–bundle (E'_G, θ') constructed above.

Proof. For Higgs vector bundles this was proved in [19] (see [19, p. 325, Theorem 5.3]). Let $(E_G\,,\theta)$ be the Higgs G–bundle defined by z in (4.9). Let $(\mathrm{ad}(E_G)\,,\varphi)$ be the corresponding Higgs vector bundle defined by the Higgs field on the adjoint vector bundle $\mathrm{ad}(E_G)$ induced by θ.

Recall that the Harder–Narasimhan reduction of the Higgs G–bundle $(E_G\,,\theta)$ is constructed using the Harder–Narasimhan filtration of the Higgs vector bundle $(\mathrm{ad}(E_G)\,,\varphi)$. Let $(E_{L(P)}\,,\theta_{L(P)})$ be the semistable principal Higgs bundle constructed as in (4.7) from the Harder–Narasimhan reduction of $(E_G\,,\theta)$. Recall that the socle reduction of a semistable Higgs $L(P)$–bundle $(E_{L(P)}\,,\theta_{L(P)})$ is constructed using the socle filtration of the adjoint vector bundle $\mathrm{ad}(E_{L(P)})$ equipped with the Higgs field induced by $\theta_{L(P)}$. From these constructions it can be deduced that the Harder–Narasimhan–socle filtration of the Higgs vector bundle $(\mathrm{ad}(E_G)\,,\varphi)$ is compatible with the filtration of $\mathrm{ad}(E_G)$ obtained from $(E_{L(P)}\,,\theta_{L(P)})$. Using this, the lemma follows. □

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