ACTION OF THE MAPPING CLASS GROUP ON CHARACTER VARIETIES AND HIGGS BUNDLES

OSCAR GARCÍA-PRADA AND GRAEME WILKIN

Abstract. We consider the action of a finite subgroup of the mapping class group $\mathrm{Mod}(S)$ of an oriented compact surface S of genus $g \geqslant 2$ on the moduli space $\mathcal{R}(S,G)$ of representations of $\pi_1(S)$ in a connected semisimple real Lie group G. Kerckhoff's solution of the Nielsen realization problem ensures the existence of an element J in the Teichmüller space of S for which Γ can be realised as a subgroup of the group of automorphisms of $X = (S, J)$ which are holomorphic or antiholomorphic. We identify the fixed points of the action of Γ on $\mathcal{R}(S,G)$ in terms of G-Higgs bundles on X equipped with a certain twisted Γ-equivariant structure, where the twisting involves abelian and non-abelian group cohomology simultaneously. These, in turn, correspond to certain representations of the orbifold fundamental group. When the kernel of the isotropy representation of the maximal compact subgroup of G is trivial, the fixed points can be described in terms of familiar objects on $Y = X/\Gamma^+$, where $\Gamma^+ \subset \Gamma$ is the maximal subgroup of Γ consisting of holomorphic automorphisms of X. If $\Gamma = \Gamma^+$ one obtains actual Γ-equivariant G-Higgs bundles on X, which in turn correspond with parabolic Higgs bundles on $Y = X/\Gamma$ (this generalizes work of Nasatyr & Steer for $G = \mathrm{SL}(2,\mathbb{R})$ and Boden, Andersen & Grove and Furuta & Steer for $G = \mathrm{SU}(n)$). If on the other hand Γ has antiholomorphic automorphisms, the objects on $Y = X/\Gamma^+$ correspond with pseudoreal parabolic Higgs bundles. This is a generalization in the parabolic setup of the pseudoreal Higgs bundles studied by the first author in collaboration with Biswas & Hurtubise.

1. Introduction

Let S be a compact oriented surface of genus greater than one, and G be a real connected semisimple Lie group. Consider the moduli space of representations or character variety $\mathcal{R}(S,G)$ defined as the space of reductive representations of the fundamental group of S in G modulo conjugation by elements of G. These are very important varieties that play a central role in geometry, topology, higher Teichmüller theory and theoretical physics (see [16] for a survey). A fundamental problem is that of understanding the action of the mapping class group or modular group of the surface $\mathrm{Mod}(S)$ in $\mathcal{R}(S,G)$. In this paper, we consider the action of a finite subgroup Γ of $\mathrm{Mod}(S)$ and give a description of the fixed-point subvariety.

A crucial step in our study is provided by a theorem of Kerckhoff solving the Nielsen realization problem [24]. This theorem proves the existence of a complex structure J on

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S, such that, if $X := (S, J)$ is the corresponding Riemann surface, Γ is a subgroup of the group of automorphisms of X which are holomorphic or antiholomorphic. We can then use holomorphic methods, and in particular the theory of G-Higgs bundles over X. To define a G-Higgs bundle, we consider a maximal compact subgroup $H \subset G$, and a Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. A G-Higgs bundle is a pair (E, φ) consisting of a $H^{\mathbb{C}}$-bundle E, where $H^{\mathbb{C}}$ is the complexification of H, and a holomorphic section φ of $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$, where $E(\mathfrak{m}^{\mathbb{C}})$ is the bundle associated to the complexification of the isotropy representation of H in $\mathfrak{m}$, and K is the canonical line bundle of X. The non-abelian Hodge correspondence establishes a homeomorphism between $\mathcal{R}(S, G)$ and the moduli space of polystable G-bundles over $X = (S, J)$ for any complex structure J on S. Now, if J is the complex structure given by Kerckhoff's theorem, using the non-abelian Hodge correspondence one can show that the action of an element of $\gamma \in \Gamma$ on $\mathcal{R}(S, G)$ coincides with the natural action of γ on $\mathcal{M}(X, G)$ via pull-back, if γ is holomorphic, or the combination of this with the conjugation defined by the reduction of the $H^{\mathbb{C}}$-bundle to H defined by the solution to the Hitchin equations, if γ is antiholomorphic. Our problem becomes then that of analysing the fixed points $\mathcal{M}(X, G)^{\Gamma}$ for this action.

The fixed-point subvariety $\mathcal{M}(X, G)^{\Gamma}$ is described in terms of G-Higgs bundles equipped with a certain twisted Γ-equivariant structure, where the twisting involves a compact conjugation τ of $H^{\mathbb{C}}$ and a group 2-cocycle $c \in Z^2_{\tau}(\Gamma, Z')$, where Z' is a τ-invariant subgroup of the centre of $H^{\mathbb{C}}$ and $\gamma \in \Gamma$ acts on $z \in Z'$ trivially if γ is holomorphic and by $\tau(z)$ if γ is antiholomorphic. We refer to this as a (Γ, τ, c)-equivariant structure. These twisted Γ-equivariant structures generalise at the same time genuine Γ-equivariant structures when Γ consists entirely of holomorphic automorphisms of X, as well as twisted real structures (referred also as pseudoreal structures in the literature) when Γ is the group generated by an antiholomorphic involution of X (see [7, 8, 9]). When Z' is contained in the kernel of the isotropy representation and Γ is a subgroup of the group of holomorphic automorphisms of X, these are lifts of true Γ-equivariant structures on the associated G/Z'-Higgs bundles.

Assuming that Γ is not a group generated by an antiholomorphic involution of X (as mentioned above, this case is treated in [7, 8, 9]), it is well-known that there is only a finite number of points $x \in X$ for which the isotropy subgroups $\Gamma_x \subset \Gamma$ for the action of Γ on X are different from the trivial subgroup $\{1\}$, and Γ_x^+, the subgroup of Γ_x consisting of holomorphic automorphisms, is a cyclic group. At such points, a (Γ, τ, c)-equivariant structure defines an element σ_x in the c_x-twisted character variety of Γ_x in $H^{\mathbb{C}}$, where $c_x \in Z^2(\Gamma_x, Z')$ is the restriction of c to Γ_x. Here $\gamma \in \Gamma_x$ acts trivially on $H^{\mathbb{C}}$ if γ is holomorphic and by τ if γ is antiholomorphic. Fixing the cocycle c and the elements σ_x at the points with $\Gamma_x \neq \{1\}$, we define the moduli space of (Γ, τ, c)-equivariant G-Higgs bundles with fixed σ_x. Our main result is Theorem 4.4, where we show that the moduli spaces of (Γ, τ, c)-equivariant G-Higgs bundles are in the fixed-point locus $\mathcal{M}(X, G)^{\Gamma}$, and more over, a smooth point in $\mathcal{M}(X, G)^{\Gamma}$ corresponds to a point in a moduli space of (Γ, τ, c)-equivariant G-Higgs bundles for some 2-cocycle c. In fact it is only the cohomology class of c which is relevant in the parametrization of fixed points. Using Theorem 4.4 and a twisted equivariant version of the non-abelian Hodge correspondence (Theorem 6.1), we describe in Theorem 6.4 the fix-point locus $\mathcal{R}(S, G)^{\Gamma}$ in terms of representations of the orbifold fundamental group for the action of Γ on S.

When Γ consists entirely of holomorphic automorphisms of X, generalising a well-known result for vector bundles [26, 15, 28, 5, 2, 1], and principal bundles [38, 3], we establish in Theorem 5.1 a correspondence between Γ-equivariant (that is, without twisting) G-Higgs bundles over X and parabolic G-Higgs bundles over $Y := X/\Gamma$. The weights of the parabolic structure are determined by the elements σ_x defined by the equivariant structure, which in this case are simply elements in the character variety $\mathrm{Hom}(\Gamma_x, H^{\mathbb{C}})/H^{\mathbb{C}}$ of Γ_x. In particular, if Z' is contained in the kernel of the isotropy representation there is a map from the moduli space of G-Higgs bundles over X to the moduli space of G/Z'-Higgs bundles and hence a map from the moduli space of (Γ, c)-equivariant G-Higgs bundles over X (here there is no twisting by τ) to the moduli space of parabolic G/Z'-Higgs bundles over Y. In this situation, using the non-abelian Hodge correspondence between parabolic G-Higgs bundles and representations of the fundamental group of a punctured surface, proved in [4], we relate in Theorem 6.5 the representations of the orbifold fundamental group for the action of Γ on S to the representations of the fundamental group of S/Γ with punctures at the points corresponding to the elements of S with non-trivial isotropy subgroup.

If we allow Γ to contain antiholomorphic automorphisms, and Γ^+ is the subgroup of Γ consisting of holomorphic automorphisms, we consider the Riemann surface $Y := X/\Gamma^+$. On this surface there is a residual antiholomorphic action of $\mathbb{Z}/2 \cong \Gamma/\Gamma^+$. Now, if the restriction of c to Γ^+ is trivial, (Γ, τ, c)-equivariant G-Higgs bundles on X are in correspondence with a pseudoreal parabolic G-Higgs bundles on Y as described in [11]. This is a generalization in the parabolic set-up of the notion of pseudoreal Higgs bundle studied in [8, 7, 6]. Again, using the non-abelian Hodge correspondence in [4], we relate in Theorem 6.6 the representations of the orbifold fundamental group for the action of Γ on S to the representations of the $\mathbb{Z}/2$-orbifold fundamental group of S/Γ^+ with punctures at the points corresponding to the elements of S with non-trivial isotropy subgroup.

The more general (Γ, τ, c)-equivariant objects on X, correspond to twisted parabolic objects on $Y := X/\Gamma^+$ in a more involved way, and will be treated in a separate paper.

In the process of writing up this paper, we came across the recent related work [33, 39, 20].

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2. Moduli space of representations and the mapping class group

In this section S is an oriented smooth compact surface of genus $g \geqslant 2$.

2.1. Moduli space of representations.

Let G be a connected real reductive Lie group. By a **representation** of $\pi_1(S)$ in G we mean a homomorphism $\rho\colon \pi_1(S) \to G$. The set of all such homomorphisms, denoted $\mathrm{Hom}(\pi_1(S), G)$, is an analytic variety, which is algebraic if G is algebraic. The group G acts on $\mathrm{Hom}(\pi_1(S), G)$ by conjugation:

$$(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1}$$

for $g \in G$, $\rho \in \mathrm{Hom}(\pi_1(S), G)$ and $\gamma \in \pi_1(S)$. If we restrict the action to the subspace $\mathrm{Hom}^+(\pi_1(S), g)$ consisting of reductive representations, the orbit space is Hausdorff. By a **reductive representation** we mean one that, composed with the adjoint representation in the Lie algebra of G, decomposes as a sum of irreducible representations. If G is algebraic this is equivalent to the Zariski closure of the image of $\pi_1(S)$ in G being a reductive group. (When G is compact every representation is reductive). The **moduli space of representations** or **character variety** of $\pi_1(S)$ in G is defined to be the orbit space

$$\mathcal{R}(S, G) = \mathrm{Hom}^+(\pi_1(S), G)/G.$$

It has the structure of an analytic variety (see e.g. [18]) which is algebraic if G is algebraic (see e.g. [30]) and is real if G is real or complex if G is complex. If G is complex then $\mathcal{R}(S, G)$ can also be expressed as the GIT quotient

$$\mathcal{R}(S, G) = \mathrm{Hom}(\pi_1(S), G) /\!\!/ G.$$

Let $\rho : \pi_1(S) \to G$ be a representation of $\pi_1(S)$ in G. Let $Z_G(\rho)$ be the centralizer in G of $\rho(\pi_1(S))$. We say that ρ is **irreducible** if and only if it is reductive and $Z_G(\rho) = Z(G)$, where $Z(G)$ is the centre of G.

2.2. **The mapping class group.** The **mapping class group** or **modular group** of S is defined as

$$\mathrm{Mod}(S) = \pi_0 \,\mathrm{Diff}(S),$$

where $\mathrm{Diff}(S)$ is the group of diffeomorphisms of S. We also consider the subgroup

$$\mathrm{Mod}^+(S) = \pi_0 \,\mathrm{Diff}^+(S),$$

where $\mathrm{Diff}^+(S)$ is the subgroup of $\mathrm{Diff}(S)$ consisting of orientation-preserving diffeomorphisms. We have an exact sequence

$$1 \to \mathrm{Mod}^+(S) \to \mathrm{Mod}(S) \to \mathbb{Z}/2 \to 1. \tag{2.1}$$

By the Dehn–Nielsen–Baer theorem, $\mathrm{Mod}(S)$ is isomorphic to $\mathrm{Out}(\pi_1(S))$, the group of outer automorphisms of $\pi_1(S)$, and hence acts in the obvious way on $\mathcal{R}(S, G)$.

Let $\Gamma \subset \mathrm{Mod}(S)$ be a finite subgroup. The main goal of this paper is to investigate the fixed points $\mathcal{R}(S, G)^\Gamma$. A crucial step to do this is provided by Kerckhoff's solution of the **Nielsen realization problem** [24]. To explain this, let J be an element in the Teichmüller space of S and $X = (S, J)$ be the corresponding Riemann surface. Denote by $\mathrm{Aut}(X)$ the group consisting of automorphisms of S which are holomorphic or antiholomorphic with respect to J. If $\mathrm{Aut}^+(X)$ is the subgroup of $\mathrm{Aut}(X)$ consisting of holomorphic automorphisms of X, there is an exact sequence

$$1 \to \mathrm{Aut}^+(X) \to \mathrm{Aut}(X) \to \mathbb{Z}/2 \to 1. \tag{2.2}$$

Theorem 2.1. *Let $\Gamma \subset \mathrm{Mod}(S)$ be a finite subgroup. There exists an element J in the Teichmüller space of S such that $\Gamma \subset \mathrm{Aut}(X)$, where $X = (S, J)$. In particular, if $\Gamma \subset \mathrm{Mod}^+(S)$, one has $\Gamma \subset \mathrm{Aut}^+(X)$. Moreover, if X is not hyperelliptic, $\Gamma = \mathrm{Aut}^+(X)$ if $\Gamma \subset \mathrm{Mod}^+(S)$, and $\Gamma = \mathrm{Aut}(X)$ if Γ is not contained in $\mathrm{Mod}^+(S)$ and $\Gamma^+ \neq \{1\}$.*

Remark 2.2. This had been proved by Nielsen [29] for cyclic groups and by Fenchel [14] for solvable groups. Thanks to Theorem 2.1 the problem of studying the action of Γ on $\mathcal{R}(S,G)$ can be reduced to studying the action of Γ on the moduli space of G-Higgs bundles on X.

2.3. **Moduli space of G-Higgs bundles.** Here X is a compact Riemann surface and G is a connected real reductive Lie group. We fix a maximal compact subgroup H of G. The Lie algebra $\mathfrak{g}$ of G is equipped with an involution θ that gives the Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H. We fix a metric B in $\mathfrak{g}$ with respect to which the Cartan decomposition is orthogonal. This metric is positive definite on $\mathfrak{m}$ and negative definite on $\mathfrak{h}$. We have $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$, $[\mathfrak{m},\mathfrak{h}] \subset \mathfrak{m}$. From the isotropy representation $H \to \mathrm{Aut}(\mathfrak{m})$, we obtain the representation $\iota : H^{\mathbb{C}} \to \mathrm{Aut}(\mathfrak{m}^{\mathbb{C}})$. When G is semisimple we take B to be the Killing form. In this case B and a choice of a maximal compact subgroup H determine a Cartan decomposition (see [25] for details).

A G-Higgs bundle on X consists of a holomorphic principal $H^{\mathbb{C}}$-bundle E together with a holomorphic section $\varphi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$, where $E(\mathfrak{m}^{\mathbb{C}})$ is the associated vector bundle with fibre $\mathfrak{m}^{\mathbb{C}}$ via the complexified isotropy representation, and K is the canonical line bundle of X.

If G is compact, $H = G$ and $\mathfrak{m} = 0$. A G-Higgs bundle is hence simply a holomorphic principal $G^{\mathbb{C}}$-bundle. If $G = H^{\mathbb{C}}$, where now H is a compact Lie group, H is a maximal compact subgroup of G, and $\mathfrak{m} = i\mathfrak{h}$. In this case, a G-Higgs bundle is a principal $H^{\mathbb{C}}$-bundle together with a section $\varphi \in H^0(X, E(\mathfrak{h}^{\mathbb{C}}) \otimes K) = H^0(X, E(\mathfrak{g}) \otimes K)$, where $E(\mathfrak{g})$ is the adjoint bundle. This is the original definition for complex Lie groups given by Hitchin in [22].

There is a notion of stability for G-Higgs bundles (see [17]). To explain this we consider the parabolic subgroups of $H^{\mathbb{C}}$ defined for $s \in i\mathfrak{h}$ as

$$P_s = \{g \in H^{\mathbb{C}} \ : \ e^{ts} g e^{-ts} \text{ is bounded as } t \to \infty\}. \tag{2.3}$$

A Levi subgroup of P_s is given by $L_s = \{g \in H^{\mathbb{C}} \ : \ \mathrm{Ad}(g)(s) = s\}$,. Their Lie algebras are given by

$$\mathfrak{p}_s = \{Y \in \mathfrak{h}^{\mathbb{C}} \ : \ \mathrm{Ad}(e^{ts})Y \text{ is bounded as } t \to \infty\},$$
$$\mathfrak{l}_s = \{Y \in \mathfrak{h}^{\mathbb{C}} \ : \ \mathrm{ad}(Y)(s) = [Y,s] = 0\}.$$

We consider the subspaces

$$\mathfrak{m}_s = \{Y \in \mathfrak{m}^{\mathbb{C}} \ : \ \iota(e^{ts})Y \text{ is bounded as } \ t \to \infty\}$$
$$\mathfrak{m}_s^0 = \{Y \in \mathfrak{m}^{\mathbb{C}} \ : \ \iota(e^{ts})Y = Y \ \text{ for every } \ t\}.$$

One has that $\mathfrak{m}_s$ is invariant under the action of P_s and $\mathfrak{m}_s^0$ is invariant under the action of L_s.

An element $s \in i\mathfrak{h}$ defines a character χ_s of $\mathfrak{p}_s$ since $\langle s, [\mathfrak{p}_s,\mathfrak{p}_s]\rangle = 0$. Conversely, by the isomorphism $(\mathfrak{p}_s/[\mathfrak{p}_s,\mathfrak{p}_s])^* \cong \mathfrak{z}_{L_s}^*$, where $\mathfrak{z}_{L_s}$ is the centre of the Levi subalgebra $\mathfrak{l}_s$, a character χ of $\mathfrak{p}_s$ is given by an element in $\mathfrak{z}_{L_s}^*$, which gives, via the invariant metric, an element of $s_\chi \in \mathfrak{z}_{L_s} \subset i\mathfrak{h}$. When $\mathfrak{p}_s \subset \mathfrak{p}_{s_\chi}$, we say that χ is an antidominant character of

$\mathfrak{p}$. When $\mathfrak{p}_s = \mathfrak{p}_{s_\chi}$ we say that χ is a strictly antidominant character. Note that for $s \in i\mathfrak{h}$, χ_s is a strictly antidominant character of $\mathfrak{p}_s$.

Let now (E, φ) be a G-Higgs bundle over X, and let $s \in i\mathfrak{h}$. Let P_s be defined as above. For $\sigma \in \Gamma(E(H^{\mathbb{C}}/P_s))$ a reduction of the structure group of E from $H^{\mathbb{C}}$ to P_s, we define the degree relative to σ and s, or equivalently to σ and χ_s in terms of the curvature of connections using Chern–Weil theory. For this, define $H_s = H \cap L_s$ and $\mathfrak{h}_s = \mathfrak{h} \cap \mathfrak{l}_s$. Then H_s is a maximal compact subgroup of L_s, so the inclusions $H_s \subset L_s$ is a homotopy equivalence. Since the inclusion $L_s \subset P_s$ is also a homotopy equivalence, given a reduction σ of the structure group of E to P_s one can further restrict the structure group of E to H_s in a unique way up to homotopy. Denote by E'_σ the resulting H_s principal bundle. Consider now a connection A on E'_σ and let $F_A \in \Omega^2(X, E'_\sigma(\mathfrak{h}_s)$ be its curvature. Then $\chi_s(F_A)$ is a 2-form on X with values in $i\mathbb{R}$, and

$$\deg(E)(\sigma, s) := \frac{i}{2\pi} \int_X \chi_s(F_A). \tag{2.4}$$

We define the subalgebra $\mathfrak{h}_{\mathrm{ad}}$ as follows. Consider the decomposition $\mathfrak{h} = \mathfrak{z} + [\mathfrak{h}, \mathfrak{h}]$, where $\mathfrak{z}$ is the centre of $\mathfrak{h}$, and the isotropy representation $\mathrm{ad} = \mathrm{ad} : \mathfrak{h} \to \mathrm{End}(\mathfrak{m})$. Let $\mathfrak{z}' = \ker(\mathrm{ad}_{|\mathfrak{z}})$ and take $\mathfrak{z}''$ such that $\mathfrak{z} = \mathfrak{z}' + \mathfrak{z}''$. Define the subalgebra $\mathfrak{h}_{\mathrm{ad}} := \mathfrak{z}'' + [\mathfrak{h}, \mathfrak{h}]$. The subindex ad denotes that we have taken away the part of the centre $\mathfrak{z}$ acting trivially via the isotropy representation ad.

Definition 2.3. *We say that a G-Higgs bundle (E, φ) is:*

semistable *if for any $s \in i\mathfrak{h}$ and any holomorphic reduction $\sigma \in \Gamma(E(H^{\mathbb{C}}/P_s))$ such that $\varphi \in H^0(X, E_\sigma(\mathfrak{m}_s) \otimes K)$, we have that* $\deg(E)(\sigma, s) \geqslant 0$*;*

stable *if for any $s \in i\mathfrak{h}_{\mathrm{ad}}$ and any holomorphic reduction $\sigma \in \Gamma(E(H^{\mathbb{C}}/P_s))$ such that $\varphi \in H^0(X, E_\sigma(\mathfrak{m}_s) \otimes K)$, we have that* $\deg(E)(\sigma, s) > 0$*;*

polystable *if it is semistable and for any $s \in i\mathfrak{h}_{\mathrm{ad}}$ and any holomorphic reduction $\sigma \in \Gamma(E(H^{\mathbb{C}}/P_s))$ such that $\varphi \in H^0(X, E_\sigma(\mathfrak{m}_s) \otimes K)$ and* $\deg(E)(\sigma, s) = 0$*, there is a holomorphic reduction of the structure group $\sigma_L \in \Gamma(E_\sigma(P_s/L_s))$ to a Levi subgroup L_s such that $\varphi \in H^0(X, E_{\sigma_L}(\mathfrak{m}^0_s) \otimes K) \subset H^0(X, E_\sigma(\mathfrak{m}_s) \otimes K)$.*

We define the **moduli space of polystable G-Higgs bundles** $\mathcal{M}(X, G)$ as the set of isomorphism classes of polystable G-Higgs bundles on X. A GIT construction of this space has been given by Schmitt [34].

The notion of stability emerges from the study of the Hitchin equations. The equivalence between the existence of solutions to these equations and the polystability of Higgs bundles is given by the following (see [17]).

Theorem 2.4. *Let (E, φ) be a G-Higgs bundle over a Riemann surface X. Then (E, φ) is polystable if and only if there exists a reduction h of the structure group of E from $H^{\mathbb{C}}$ to H, such that*

$$F_h - [\varphi, \tau_h(\varphi)] = 0 \tag{2.5}$$

where $\tau_h : \Omega^{1,0}(E(\mathfrak{m}^{\mathbb{C}})) \to \Omega^{0,1}(E(\mathfrak{m}^{\mathbb{C}}))$ is the combination of the anti-holomorphic involution in $E(\mathfrak{m}^{\mathbb{C}})$ defined by the compact real form at each point determined by h and the *conjugation of* 1*-forms, and* F_h *is the curvature of the unique* H*-connection compatible with the holomorphic structure of* E*.*

A G-Higgs bundle (E,φ) is said to be **simple** if $\mathrm{Aut}(E,\varphi) = Z(H^{\mathbb{C}}) \cap \ker(\iota)$ where $Z(H^{\mathbb{C}})$ the centre of $H^{\mathbb{C}}$. A G-Higgs bundle (E,φ) is said to be **infinitesimally simple** if the infinitesimal automorphism space $\mathrm{aut}(E,\varphi)$ is isomorphic to $H^0(X, E(\ker d\iota \cap Z(\mathfrak{h}^{\mathbb{C}}))$ where $Z(\mathfrak{h}^{\mathbb{C}})$ denotes the Lie algebra of $Z(H^{\mathbb{C}})$.

Thus a G-Higgs bundle is (infinitesimally) simple if its (infinitesimal) automorphism group is as small as possible. It is clear that a simple G-Higgs bundle is infinitesimally simple. If G is complex then ι is the adjoint representation and (E,φ) is simple (resp. infinitesimally simple) if $\mathrm{Aut}(E,\varphi) = Z(G)$ (resp. $\mathrm{aut}(E,\varphi) = Z(\mathfrak{g})$).

The basic link between representations of $\pi_1(S)$ and Higgs bundles is given by the **non-abelian Hodge correspondence** due to Hitchin, Donaldson, Simpson, Corlette and others (see [17] and references there).

Theorem 2.5. *Let* S *be a compact surface and* $X = (S, J)$ *be the Riemann surface defined by any complex structure* J *on* S*. Let* G *be a real connected semisimple Lie group. There is a homeomorphism* $\mathcal{R}(S,G) \xrightarrow{\cong} \mathcal{M}(X,G)$*, where the image of the irreducible representations is the subspace of stable and simple* G*-Higgs bundles.*

A key step to go from a polystable G-Higgs bundle (E,φ) over X to a representation ρ of $\pi_1(S)$ in G is given by the relation

$$\nabla = \bar{\partial}_E - \tau_h(\bar{\partial}_E) + \varphi - \tau_h(\varphi), \tag{2.6}$$

where ∇ is the flat connection corresponding to ρ, $\bar{\partial}_E$ is the Dolbeault operator of E and τ_h is provided by the solution to the Hitchin equations in Theorem 2.4. The converse construction is provided by the Donaldson–Corlette theorem on the existence of harmonic metrics on a reductive flat bundle given in [13, 12].

Remark 2.6*.* Theorem 2.5 can also be extended to (non-connected) reductive groups. The presence of a continuous centre in G requires replacing the fundamental group of S by its universal central extension.

From Theorems 2.1 and 2.5 we conclude the following.

Proposition 2.7. *Let* $\Gamma \subset \mathrm{Mod}(S)$ *be a finite subgroup and* $\Gamma^+ = \Gamma \cap \mathrm{Mod}^+(S)$*. Let* J *be a complex structure given by Kerckhoff's theorem and* $X = (S, J)$ *be the corresponding Riemann surface. Under the non-abelian Hodge correspondence* $\mathcal{R}(S,G) \cong \mathcal{M}(X,G)$ *given by Theorem 2.5, the action of* Γ *on* $\mathcal{R}(S,G)$ *coincides with the following action of* Γ *on* $\mathcal{M}(X,G)$*:*

$$\gamma \cdot (E,\varphi) = \begin{cases} (\gamma^* E, \gamma^*\varphi) & \text{if } \gamma \in \Gamma^+ \\ (\gamma^*\tau_h(E), \gamma^*\tau_h(\varphi)) & \text{if } \gamma \notin \Gamma^+ \end{cases}$$

where τ_h *is given by Theorem 2.4,* $\tau_h(E) := E \times_{\tau_h} (H^{\mathbb{C}})$ *and* $\tau_h(\varphi)$ *is as in Theorem 2.4. We thus have that for this action* $\mathcal{R}(S,G)^\Gamma$ *and* $\mathcal{M}(X,G)^\Gamma$ *are in bijective correspondence.*

Proof. Given any $\gamma \in \Gamma \subset \mathrm{Mod}(S)$, Kerckhoff's theorem [24, Thm. 5] guarantees a unique diffeomorphism f in the isotopy class of γ such that $f^*J = J$ if $\gamma \in \Gamma^+$ or $f^*J = -J$ if $\gamma \notin \Gamma^+$. The action of γ on $\mathcal{R}(S,G)$ is defined by $\gamma \cdot [\rho] = [f^*\rho] = [\rho \circ f_*]$, which induces an action on the space of equivalence classes of flat connections given by $\gamma \cdot [\nabla] = [f^*\nabla]$ if $\gamma \in \Gamma^+$ or $\gamma \cdot [\nabla] = [-f^*\nabla]$ if $\gamma \notin \Gamma^+$. To find the induced action of γ on $\mathcal{M}(X,G)$ via Theorem 2.5 (which is well-defined since $f^*J = \pm J$) we recall that the flat connection ∇ associated to a polystable G-Higgs bundle (E,φ) is given by (2.6), and observe that $\tau_h(\bar\partial_E)$ is the Dolbeault operator of $\tau_h(E)$. Thus proving the statement. □

3. Twisted equivariant structures on principal bundles and associated vector bundles

In this section X is a compact Riemann surface of genus bigger than one, $\Gamma \subset \mathrm{Aut}(X)$, G is a connected **complex** reductive Lie group, and τ is a conjugation of G (not necessarily the compact conjugation). We will write $\Gamma = \Gamma^+ \cup \Gamma^-$, where Γ^+ is the subgroup of Γ consisting of holomorphic automorphisms and Γ^- is the coset consisting of antiholomorphic automorphisms.

3.1. Twisted equivariant structures on a principal bundle.

Let $Z := Z(G)$ be the centre of G. Consider the action of τ on Z and let $Z' \subset Z$ be a subgroup invariant under this action. Consider the action of Γ on Z' given by

$$z^\gamma = \begin{cases} z & \text{if } \gamma \in \Gamma^+ \\ \tau(z) & \text{if } \gamma \in \Gamma^-. \end{cases} \tag{3.1}$$

Let $c \in Z^2_\tau(\Gamma, Z')$ be a 2-**cocycle** for this action. This is a map $c : \Gamma \times \Gamma \to Z'$ satisfying the cocycle condition

$$c(\gamma', \gamma'')^\gamma c(\gamma, \gamma'\gamma'') = c(\gamma\gamma', \gamma'')c(\gamma, \gamma').$$

These objects emerge in the study of "lifts" to G of **non-abelian 1-cocycles** in $Z^1(\Gamma, G/Z')$ for the action of Γ on G given by $g^\gamma = g$ if γ is holomorphic and $g^\gamma = \tau(g)$ if γ is anti-holomorphic. In particular, if $\Gamma = \Gamma^+$, the action of Γ on G is trivial and $Z^1(\Gamma, G/Z') = \mathrm{Hom}(\Gamma, G/Z')$, that is the 1-cocycles are simply representations of Γ in G/Z'.

Let E be a holomorphic G-bundle over X. Let $c \in Z^2_\tau(\Gamma, Z')$. A (Γ, τ, c)-**equivariant structure on** E (or simply **twisted** Γ-**equivariant structure** if there is no need to specify τ and c) consists of a collection of maps $\widetilde{\gamma} : E \to E$ covering $\gamma : X \to X$ for every $\gamma \in \Gamma$, satisfying

$$\widetilde{\gamma}(eg) = \begin{cases} \widetilde{\gamma}(e)g \ \text{ and } \ \gamma \ \text{ holomorphic} & \text{if } \gamma \in \Gamma^+ \\ \widetilde{\gamma}(e)\tau(g) \ \text{ and } \ \gamma \ \text{ antiholomorphic} & \text{if } \gamma \in \Gamma^-, \end{cases}$$

$$\widetilde{\gamma\gamma'} = c(\gamma, \gamma')\widetilde{\gamma}\widetilde{\gamma}',$$

and $\widetilde{\mathrm{Id}_X} = \mathrm{Id}_E$. This imposes the condition $c(\gamma, 1) = 1$ for every $\gamma \in \Gamma$.

When c is the trivial cocycle 1 we will refer to a $(\Gamma, \tau, 1)$-equivariant structure as a (Γ, τ)-equivariant structure or a τ-twisted Γ-equivariant structure. If $\Gamma = \Gamma^+$, we take τ

to be the identity and we refer to a $(\Gamma, 1, c)$-equivariant structure as a (Γ, c)-equivariant structure. If, moreover $c = 1$, then we obtain a genuine Γ-equivariant structure on E.

Let $\mathrm{Aut}(E)$ be the group of holomorphic automorphisms of E covering the identity of X, and let $\mathrm{Aut}_{\Gamma,\tau}(E)$ be the group of bijective maps $f : E \to E$ defined by

$$f(eg) = \begin{cases} f(e)g \ \text{ and } \ f \ \text{ holomorphic} & \text{if } \ f \ \text{ covers } \ \gamma \in \Gamma^+ \\ f(e)\tau(g) \ \text{ and } \ f \ \text{ antiholomorphic} & \text{if } \ f \ \text{ covers } \ \gamma \in \Gamma^-. \end{cases} \tag{3.2}$$

There is an exact sequence

$$1 \to \mathrm{Aut}(E) \to \mathrm{Aut}_{\Gamma,\tau}(E) \to \Gamma. \tag{3.3}$$

A (Γ, τ, c)-equivariant structure on E is simply a twisted representation $\Gamma \to \mathrm{Aut}_{\Gamma,\tau}(E)$ with cocycle c, that is a map $\sigma : \Gamma \to \mathrm{Aut}_{\Gamma,\tau}(E)$ such that

$$\sigma(\gamma\gamma') = c(\gamma, \gamma')\sigma(\gamma)\sigma(\gamma').$$

This is clear since, if E' is the G/Z'-principal bundle associated to E via the projection $G \to G/Z'$, a (Γ, τ, c)-equivariant structure on E defines a (Γ, τ)-equivariant structure on E', and we have an exact sequence

$$1 \to Z' \to \mathrm{Aut}_{\Gamma,\tau}(E) \to \mathrm{Aut}_{\Gamma,\tau}(E') \to 1.$$

Two twisted Γ-equivariant structures on E for the same τ and for two cocycles c and c' define the same (Γ, τ)-equivariant structure on E' if and only if there exists a function $f : G \to Z'$ such that the corresponding twisted representations σ and σ' of Γ in $\mathrm{Aut}_{\Gamma,\tau}(E)$ are related by $\sigma' = f\sigma$, and

$$c'(\gamma, \gamma') = f(\gamma\gamma')f(\gamma)^{-1}f(\gamma')^{-1}c(\gamma, \gamma'). \tag{3.4}$$

This defines a natural equivalence relation in the set of (Γ, τ, c)-equivariant structures on E, whose equivalence classes are parametrised by the cohomology group $H^2(\Gamma, Z')$.

Remark 3.1. Of course if $Z' = Z$, $G/Z' = \mathrm{Ad}(G)$ and $E' = P(E) := E/Z$.

There is an alternative way of thinking of a (Γ, τ, c)-equivariant structure as a τ-twisted equivariant structure on E for the action of a larger group. Namely, the 2-cocycle c defines an extension of groups

$$1 \to Z' \to \Gamma_c \to \Gamma \to 1.$$

Two cocycles are cohomologous if and only if the corresponding extensions are equivalent, i.e. equivalence classes of extensions of Γ by Z' with the action of Γ on Z' given by 3.1 are parametrised by $H^2_\tau(\Gamma, Z')$.

We have the following.

Proposition 3.2. *(Γ, τ, c)-equivariant structures on E are in bijection with central (Γ_c, τ)-equivariant structures on E, where Γ_c acts on X and on Z' via the projection $\Gamma_c \to \Gamma$, and by central we mean that the action of Z' in the kernel of the extension above is the natural action of Z' on E.*

Proof. It follows from group representation theory (see [31] for example) that a twisted representation $\Gamma \to \mathrm{Aut}_{\Gamma,\tau}(E)$ with cocycle c is equivalent to a representation $\rho : \Gamma_c \to \mathrm{Aut}_{\Gamma_c,\tau}(E)$ fitting in the following commutative diagram, where $\widetilde{\rho}$ is the induced representation

$$\begin{tikzcd}
0 \arrow[r] & Z' \arrow[r] \arrow[d, "\mathrm{Id}"'] & \Gamma_c \arrow[r] \arrow[d, "\rho"'] & \Gamma \arrow[r] \arrow[d, "\tilde{\rho}"'] & 1 \\
1 \arrow[r] & Z' \arrow[r] & \mathrm{Aut}_{\Gamma_c,\tau}(E) \arrow[r] & \mathrm{Aut}_{\Gamma,\tau}(E') \arrow[r] & 1.
\end{tikzcd}$$

This completes the proof. □

Recall that a G-bundle E is said to be **simple** if $\mathrm{Aut}(E) \cong Z$. We have the following.

Proposition 3.3. *Let E be a simple G-bundle over X such that*

$$E \cong \begin{cases} \gamma^* E & \text{for every } \gamma \in \Gamma^+ \\ \gamma^* \tau(E) & \text{for every } \gamma \in \Gamma^-. \end{cases} \tag{3.5}$$

Then E admits a (Γ, τ, c)-equivariant structure with $c \in Z^2_\tau(\Gamma, Z)$.

Proof. Condition (3.5) implies the existence of an exact sequence

$$1 \to \mathrm{Aut}(E) \to \mathrm{Aut}_{\Gamma,\tau}(E) \to \Gamma \to 1.$$

Now, since E is simple $\mathrm{Aut}(E) \cong Z$ and hence we have an extension

$$1 \to Z \to \mathrm{Aut}_{\Gamma,\tau}(E) \to \Gamma \to 1.$$

This extension is determined by a cocycle $c \in Z^2_\tau(\Gamma, Z)$, which is precisely the obstruction to having a (Γ, τ)-equivariant structure on E, i.e. a homomorphism $\Gamma \to \mathrm{Aut}_{\Gamma,\tau}(E)$ splitting the exact sequence. However we have a twisted homomorphism of Γ in $\mathrm{Aut}_{\Gamma,\tau}(E)$ with cocycle c, that is, a (Γ, τ, c)-equivariant structure. □

3.2. **Isotropy subgroups associated to a (Γ, τ, c)-equivariant structure.** We will assume that $\Gamma^+ \neq \{1\}$. The case $\Gamma^+ = \{1\}$ has been extensively studied in [7, 8, 9] and corresponds to the study of twisted real structures on E.

Let $x \in X$, and

$$\Gamma_x := \{\gamma \in \Gamma^+ \ : \ \gamma(x) = x\}$$

be the corresponding **isotropy subgroup**. Let $\mathscr{P} = \{x \in X \ : \ \Gamma_x \neq \{1\}\}$.

The following is well-known (see [28] for example).

Proposition 3.4. *(1) $\mathscr{P}$ consists of a finite number of points $\{x_1, \ldots, x_r\} \subset X$.*

(2) For each $x_i \in \mathscr{P}$, Γ_{x_i} is cyclic.

Let $c_x \in Z^2(\Gamma_x, Z')$ be the restriction of c to Γ_x (note that the action of Γ_x on Z' is trivial since $\Gamma_x \subset \Gamma^+$). Define the c_x-**twisted character variety** of Γ_x in G as the set

$$R_{c_x}(\Gamma_x, G) := \mathrm{Hom}_{c_x}(\Gamma_x, G)/G,$$

where

$$\mathrm{Hom}_{c_x}(\Gamma_x, G) := \{\sigma : \Gamma_x \to G \;\mid\; \sigma(\gamma\gamma') = c_x(\gamma,\gamma')\sigma(\gamma)\sigma(\gamma')\},$$

and two elements $\sigma, \sigma' \in \mathrm{Hom}_{c_x}(\Gamma_x, G)$ are equivalent under the action of G if

$$\sigma'(\gamma) = g^{-1}\sigma(\gamma)g \;\text{ for some }\; g \in G.$$

Proposition 3.5. *A (Γ,τ,c)-equivariant structure on a G-bundle $\pi : E \to X$ defines for every $x \in \mathscr{P}$ an element $\sigma_x \in R_{c_x}(\Gamma_x, G)$.*

Proof. For each $x \in \mathscr{P}$ and $e \in E$ such that $\pi(e) = x$, a straightforward computation shows that the map $\sigma_e : \Gamma_x \to G$ given by

$$\widetilde{\gamma}(e) = e\sigma_e(\gamma) \tag{3.6}$$

defines an element in $\mathrm{Hom}_{c_x}(\Gamma_x, G)$. Moreover, if $e' \in \pi^{-1}(x)$, with $e' = eg$ for $g \in G$, then $\sigma_{e'}(\gamma) = g^{-1}\sigma_e(\gamma)g$, proving the assertion. □

Remark 3.6. The composition of σ_e with the projection $G \to G/Z'$, defines a homomorphism $\rho_e : \Gamma_x \to G/Z'$. Of course, c restricted to Γ^+ is trivial, i.e., if the restriction of the action of Γ to Γ^+ defines a genuine Γ^+-equivariant structure on E, then σ_e itself is a homomorphism, and σ_x is an element of the character variety $R(\Gamma_x, G) := \mathrm{Hom}(\Gamma_x, G)/G$.

The following is clear.

Proposition 3.7. *Let c and c' be 2-cocycles in $Z^2_\tau(\Gamma, Z')$. Let $\sigma_x \in R_{c_x}(\Gamma_x, G)$ and $\sigma'_x \in R_{c'_x}(\Gamma_x, G)$ be corresponding classes. Then the projections of σ_x and σ'_x in $R(\Gamma_x, G/Z')$ coincide.*

The next result shows that the Γ action defines a bijection between spaces of twisted representations of isotropy groups over points in X related by the action of Γ.

Proposition 3.8. *(1)The action of Γ on X induces an action of Γ (and in particular of Γ^+) on $\mathscr{P}$.*

(2) Let $\mathscr{Q} = \mathscr{P}/\Gamma^+$. If x and x' are in the same class in $\mathscr{Q}$ there is an isomorphism $R_{c_x}(\Gamma_x, G) \cong R_{c_{x'}}(\Gamma_{x'}, G)$ (as pointed sets) under which σ_x and $\sigma_{x'}$ are in correspondence. This isomorphism induces a canonical isomorphism $R(\Gamma_x, G/Z') \cong R(\Gamma_{x'}, G/Z')$.

(3) If two points $y, y' \in \mathscr{Q}$ are in correspondence under the residual action of $\mathbb{Z}/2 \cong \Gamma/\Gamma^+$ on $\mathscr{Q}$, then $R_{c_x}(\Gamma_x, G)$ and $R_{c_{x'}}(\Gamma_{x'}, G)$ are in a bijective correspondence given by $\sigma_x \mapsto \tau\sigma_{x'}$ for any representatives $x, x' \in \mathscr{P}$ of $y, y' \in \mathscr{Q}$ respectively.

Proof. Statement (1) follows from the fact that two points on X connected by the action of Γ must have conjugate isotropy subgroups. To prove (2), if two points $x, x' \in \mathscr{P}$ are in the same class in $\mathscr{Q}$, then there exists $\gamma_0 \in \Gamma^+$ such that $x' = \gamma_0 \cdot x$ and so $\Gamma_{x'} = \gamma_0\Gamma_x\gamma_0^{-1}$. Let $\tilde{\gamma}_0$ denote the lift of γ_0 to $\mathrm{Aut}_{\Gamma^+}(E)$, where $\mathrm{Aut}_{\Gamma^+}(E)$ is the preimage of Γ^+ in the exact sequence (3.3). Given any e_x in the fibre E_x, let $e_{x'} := \tilde{\gamma}_0(e_x)$. For any $\gamma \in \Gamma_x$, let $\gamma' = \gamma_0\gamma\gamma_0^{-1}$ be the corresponding element of $\Gamma_{x'}$ and let $\tilde{\gamma}$ and $\tilde{\gamma}' = \tilde{\gamma}_0\tilde{\gamma}\tilde{\gamma}_0^{-1}$ denote the respective lifts to $\mathrm{Aut}_\Gamma(E)$. Using (3.6) we have

$$\tilde{\gamma}(e_x) = e_x\sigma_{e_x}(\gamma) \quad\text{and}\quad \tilde{\gamma}'(e_{x'}) = e_{x'}\sigma_{e_{x'}}(\gamma').$$

Therefore

$$e_{x'}\sigma_{e_{x'}}(\gamma') = \tilde{\gamma}'(e_{x'}) = \tilde{\gamma}_0\tilde{\gamma}\tilde{\gamma}_0^{-1}(e_{x'}) = \tilde{\gamma}_0\tilde{\gamma}(e_x) = \tilde{\gamma}_0 e_x\sigma_{e_x}(\gamma) = e_{x'}\sigma_{e_x}(\gamma)$$

and so $\sigma_{e_{x'}}(\gamma') = \sigma_{e_{x'}}(\gamma_0\gamma\gamma_0^{-1}) = \sigma_{e_x}(\gamma)$. Therefore we see that σ_{e_x} determines $\sigma_{e_{x'}}$ and vice versa, and so the same is true for σ_x and $\sigma_{x'}$.

Therefore, a choice of γ_0 such that $x' = \gamma_0 \cdot x$ determines a bijection $R_{c_x}(\Gamma_x, G) \to R_{c_{x'}}(\Gamma_{x'}, G)$ sending $\sigma \mapsto \sigma'$ with $\sigma'(\gamma') := \sigma(\gamma_0^{-1}\gamma'\gamma_0)$, and this bijection maps σ_x to $\sigma_{x'}$.

An element $\sigma \in \mathrm{Hom}_{c_x}(\Gamma_x, G)$ descends to a homomorphism $\bar{\sigma} : \Gamma_x \to G/Z'$. The bijection $\sigma \mapsto \sigma'$ defined above induces a map $\bar{\sigma} \mapsto \bar{\sigma}'$ defined by

$$\bar{\sigma}'(\gamma') := \bar{\sigma}(\gamma_0^{-1}\gamma'\gamma_0).$$

Given any other choice γ_1 such that $\Gamma_{x'} = \gamma_1\Gamma_x\gamma_1^{-1}$, we have $\gamma_1\gamma_0^{-1} \in \Gamma_{x'}$ and so (since $\bar{\sigma}$ is a homomorphism) for any $\gamma' \in \Gamma_{x'}$ we have

$$\bar{\sigma}(\gamma_1^{-1}\gamma'\gamma_1) = \bar{\sigma}(\gamma_1^{-1}\gamma_0)\bar{\sigma}(\gamma_0^{-1}\gamma'\gamma_0)\bar{\sigma}(\gamma_1^{-1}\gamma_0)^{-1}.$$

Therefore the conjugacy class of $\bar{\sigma}'$ in $R(\Gamma_{x'}, G/Z')$ is well-defined and independent of the choice of γ_0 such that $\Gamma_{x'} = \gamma_0\Gamma_x\gamma_0^{-1}$.

(3) follows from a straightforward computation.

□

Remark 3.9. Note that if in (3) $y \in \mathscr{Q}$ is a fixed point under the residual action of $\mathbb{Z}/2$ then the twisted representation σ_x must lie in the real group G^τ.

3.3. **Twisted Γ-equivariant structures on associated vector bundles.** Let now V be a rank n holomorphic complex vector bundle over X. Let $\tau_{\mathbb{V}}$ be a conjugation on the fibre $\mathbb{V}$ of V. Consider the action of Γ on $\mathbb{C}^*$ given by

$$z^\gamma = \begin{cases} z & \text{if } \gamma \in \Gamma^+ \\ \bar{z} & \text{if } \gamma \in \Gamma^-. \end{cases} \tag{3.7}$$

Given a cocycle $c \in Z^2(\Gamma, \mathbb{C}^*)$ for this action, similarly as for G-bundles one can define a $(\Gamma, \tau_{\mathbb{V}}, c)$-equivariant structure on V as a c-twisted representation of Γ in $\mathrm{Aut}_{\Gamma,\tau_{\mathbb{V}}}(V)$, where $\mathrm{Aut}_{\Gamma,\tau_{\mathbb{V}}}(V)$ is defined in a similar fashion to the G-bundle case.

Now, let E be a principal G-bundle and $\rho : G \to GL(\mathbb{V})$ a representation of G in a complex vector space $\mathbb{V}$. Consider the associated vector bundle $V := E(\mathbb{V})$. Let τ and $\tau_{\mathbb{V}}$ be conjugations of G and $\mathbb{V}$, respectively Let $c \in Z^2_\tau(\Gamma, Z')$ and $c_\rho \in Z^2(\Gamma, \mathbb{C}^*)$ be the cocycle induced by $\rho|_{Z'} : Z' \to \mathbb{C}^* \cong Z(\mathrm{GL}(\mathbb{V}))$. If ρ is compatible with the conjugations τ and $\tau_{\mathbb{V}}$, then there is a homomorphism $\mathrm{Aut}_{\Gamma,\tau}(E) \to \mathrm{Aut}_{\Gamma,\tau_{\mathbb{V}}}(V)$, and it is clear that a (Γ, τ, c)-equivariant structure on E defines a $(\Gamma, \tau_{\mathbb{V}}, c_\rho)$-equivariant structure on V. In particular if $Z' \subset \ker\rho$, then c_ρ is trivial and hence we obtain $(\Gamma, \tau_{\mathbb{V}})$-equivariant structure on V. If moreover $\Gamma = \Gamma^+$, this is a genuine Γ-equivariant structure on V.

4. Twisted equivariant structures on Higgs bundles

In this section X is a compact Riemann surface of genus bigger than one, Γ is a subgroup of $\mathrm{Aut}(X)$, the group of holomorphic or antiholomorphic automorphisms of X, and G

is a connected **real** reductive Lie group. As in Section 2.3, we fix a maximal compact subgroup H of G. The Lie algebra $\mathfrak{g}$ of G is equipped with an involution θ that gives the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H. We choose a complex conjugation τ of $H^{\mathbb{C}}$, and a conjugation $\tau_{\mathfrak{m}^{\mathbb{C}}}$ of $\mathfrak{m}^{\mathbb{C}}$, such that the isotropy representation $\iota : H^{\mathbb{C}} \to \mathrm{Aut}(\mathfrak{m}^{\mathbb{C}})$ is compatible with τ and $\tau_{\mathfrak{m}^{\mathbb{C}}}$. This is the case if for example G is a real form of a complex reductive group $G^{\mathbb{C}}$ and we choose a complex conjugation $\widetilde{\tau}$ of $G^{\mathbb{C}}$ commuting with the Cartan involution of G extended to $G^{\mathbb{C}}$. The conjugation τ and $\tau_{\mathfrak{m}^{\mathbb{C}}}$ induced by $\widetilde{\tau}$ satisfy the compatibility condition with ι. As proved by Cartan, we can always choose a compact conjugation $\widetilde{\tau}$ commuting with the Cartan involution. This is the choice which is relevant in connection to the study of Γ on the moduli space of representations $\mathcal{R}(S, G)$.

4.1. **Twisted Γ-equivariant structures on G-Higgs bundles.** Let (E, φ) be a G-Higgs bundle over X. We will define now twisted Γ-equivariant structures on (E, φ). To do this, let $Z = Z(H^{\mathbb{C}})$, and let $Z' \subset Z$ be a subgroup invariant under the action of τ. Choose a 2-cocycle $c \in Z^2_\tau(\Gamma, Z')$. Recall from Section 3.3, that this defines a 2-cocycle $c_\iota \in Z^2(\Gamma, \mathbb{C}^*)$, via the isotropy representation $\iota : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$.

If the $H^{\mathbb{C}}$-bundle E is equipped with (Γ, τ, c)-equivariant structure, from Section 3.3, the vector bundle $E(\mathfrak{m}^{\mathbb{C}})$ inherits a $(\Gamma, \tau_{\mathfrak{m}^{\mathbb{C}}}, c_\iota)$-equivariant structure. On the other hand, the canonical bundle K over X has a natural $(\Gamma, \tau_{\mathbb{C}})$-equivariant structure induced by the action of Γ on X. We conclude then that the bundle $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$ has a $(\Gamma, \tau_{\mathfrak{m}^{\mathbb{C}}}, c_\iota)$-equivariant structure (where we are omitting $\tau_{\mathbb{C}}$ in the notation). In fact, we will abuse notation, and use τ to refer to both τ and $\tau_{\mathfrak{m}^{\mathbb{C}}}$ in the sequel.

A (Γ, τ, c)**-equivariant structure on** (E, φ) is a (Γ, τ, c)-equivariant structure on E, such that for every $\gamma \in \Gamma$ the following diagram commutes:

$$\begin{array}{ccc} E(\mathfrak{m}^{\mathbb{C}}) \otimes K & \xrightarrow{\widetilde{\gamma}} & E(\mathfrak{m}^{\mathbb{C}}) \otimes K \\ \Big\uparrow \varphi & & \Big\uparrow \varphi \\ X & \xrightarrow{\gamma} & X \end{array},$$

where $\widetilde{\gamma}$ is the collection of maps defining the (Γ, τ, c_ι)-equivariant structure on $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$ defined above.

The notion of stability for G-Higgs bundles given in Section 2.3 (see [17]) can be extended in a natural way to a G-Higgs bundle equipped with a twisted equivariant structure. Everything is exactly the same except that the reductions to the parabolic subgroups $P_s \subset H^{\mathbb{C}}$ have to be Γ-invariant. For this it is important to observe that a twisted equivariant structure on E defines actually a Γ action on the space of reductions of E to P_s. In fact we have the following more general result.

Lemma 4.1. *Let E be a $H^{\mathbb{C}}$-bundle over X equipped with a (Γ, τ, c)-equivariant structure, where $c \in Z^2_\tau(\Gamma, Z')$, and let $H' \subset H^{\mathbb{C}}$ be a subgroup such that $Z' \subset H'$. Then the twisted equivariant structure on E induces a a group action of Γ on the space of reductions of E to H'.*

Proof. Recall that a reduction of E to H' is a section of $E(H^{\mathbb{C}}/H')$, the $H^{\mathbb{C}}/H'$-bundle associated to E via the natural action of $H^{\mathbb{C}}$ on $H^{\mathbb{C}}/H'$ on the left. Such a section is equivalent to a map $\psi : E \to H^{\mathbb{C}}/H'$ such that $\psi(eh) = h^{-1}e$, for every $e \in E$ and $h \in H^{\mathbb{C}}$. Now, let $\gamma \in \Gamma$ and define

$$\gamma \cdot \psi(e) := \psi(\widetilde{\gamma}(e)),$$

where $\widetilde{\gamma}$ is given by the (Γ, τ, c)-equivariant structure on E. We need to check that $(\gamma\gamma') \cdot \psi = \gamma \cdot (\gamma' \cdot \psi)$ for every $\gamma, \gamma' \in \Gamma$. We have

$$((\gamma\gamma') \cdot \psi)(e) = \psi(\widetilde{\gamma\gamma'}(e)) = \psi(c(\gamma,\gamma')\widetilde{\gamma}(\widetilde{\gamma'}(e))) = c(\gamma,\gamma')^{-1}\psi(\widetilde{\gamma}(\widetilde{\gamma'}(e)))$$

but since $Z' \subset H'$

$$c(\gamma,\gamma')^{-1}\psi(\widetilde{\gamma}(\widetilde{\gamma'}(e))) = \psi(\widetilde{\gamma}(\widetilde{\gamma'}(e)) = (\gamma \cdot (\gamma' \cdot \psi))(e).$$

□

To define the moduli space of twisted Γ-equivariant G-Higgs bundles, we fix the cocycle $c \in Z^2_\tau(\Gamma, Z')$ and the elements $\sigma_i \in R_{c_{x_i}}(\Gamma_{x_i}, H^{\mathbb{C}})$ for every point $x_i \in \mathscr{P}$ defined by Proposition 3.5. We will need at some point the projection of σ_i in $R(\Gamma_{x_i}, H^{\mathbb{C}}/Z')$ that we will denote by $[\sigma_i]$. Let $\sigma = (\sigma_1, \cdots, \sigma_r)$. We define $\mathcal{M}(X, G, \Gamma, \tau, c)$ to be the **moduli space of polystable** (Γ, τ, c)**-equivariant** G**-Higgs bundles**. An analytic construction of these spaces can be given using slices. The subvariety of $\mathcal{M}(X, G, \Gamma, \tau, c)$ with fixed classes σ will be denoted by $\mathcal{M}(X, G, \Gamma, \tau, c, \sigma)$.

Given a (Γ, τ, c)-equivariant G-Higgs bundle (E, φ) such that $Z' \subset H$, by Lemma 4.1 we consider the action of Γ on the space of metrics on E, that is on the space of sections of $E(H^{\mathbb{C}}/H)$. The analysis done for the Hitchin–Kobayashi correspondence given in Section 2.3 can be extended to this equivariant situation to prove the following (see [6]).

Theorem 4.2. *Let (E, φ) be a G-Higgs bundle over a Riemann surface X equipped with a (Γ, c)-equivariant structure, with cocycle $c \in Z^2(\Gamma, Z')$ and $Z' \subset H$. Then (E, φ) is polystable as a (Γ, c)-equivariant Higgs bundle if and only if there exists a Γ-invariant reduction h of the structure group of E from $H^{\mathbb{C}}$ to H, such that*

$$F_h - [\varphi, \tau_h(\varphi)] = 0. \tag{4.1}$$

From Theorems 4.2 and 2.4 we conclude the following.

Proposition 4.3. *Let $Z' \subset Z \cap H$ and $c \in Z^2(\Gamma, Z')$. Then the forgetful map defines a morphism $\mathcal{M}(X, G, \Gamma, \tau, c, \sigma) \to \mathcal{M}(X, G)$.*

4.2. **Γ-action on the moduli space of G-Higgs bundles.** Consider the action of Γ on the moduli space of G-Higgs bundles $\mathcal{M}(X, G)$ given by the rule:

$$\gamma \cdot (E, \varphi) = \begin{cases} (\gamma^* E, \gamma^* \varphi) & \text{if } \gamma \in \Gamma^+ \\ (\gamma^* \tau(E), \gamma^* \tau(\varphi)) & \text{if } \gamma \notin \Gamma^+. \end{cases}$$

We have the following.

Theorem 4.4. *Let $Z' \subset Z \cap H$ and $\widetilde{\mathcal{M}}(X, G, \Gamma, \tau, c, \sigma)$ be the image of the morphism in Proposition 4.3. Then*

(1) If c and c' are cohomologous cocycles in $Z^2_\tau(\Gamma, Z')$

$$\widetilde{\mathcal{M}}(X, G, \Gamma, \tau, c, \sigma) = \widetilde{\mathcal{M}}(X, G, \Gamma, \tau, c', \sigma').$$

(2) For any $Z' \subset Z$, any σ, and any cocycle $c \in Z^2_\tau(\Gamma, Z')$

$$\widetilde{\mathcal{M}}(X, G, \Gamma, \tau, c, \sigma) \subset \mathcal{M}(X, G)^\Gamma.$$

(3) Let $\mathcal{M}_*(X, G) \subset \mathcal{M}(X, G)$ be the subvariety of G-Higgs bundles which are stable and simple and let $Z' = Z \cap \ker \iota$, then

$$\mathcal{M}(X, G)^\Gamma_* \subset \bigcup_{[c] \in H^2_\tau(\Gamma, Z'), \sigma : [\sigma_i] \in R(\Gamma_{x_i}, H^{\mathbb{C}}/Z')} \widetilde{\mathcal{M}}(X, G, \Gamma, \tau, c, \sigma).$$

Proof. To prove (1), we consider the function $f : G \to Z'$ such that c and c' are related by (3.4). This function defines an automorphism of a G-Higgs bundle (E, φ) which sends the twisted equivariant structure with cocycle c and isotropy σ to a twisted equivariant structure with cocycle c' and isotropy σ'. The proof of (2) follows immediately from the definition of twisted equivariant structure. The proof of (3) follows a similar argument to that of Proposition 3.3: The condition $(E, \varphi) \cong (\gamma^* E, \gamma^* \varphi)$ if $\gamma \in \Gamma^+$ or $(E, \varphi) \cong (\gamma^* \tau(E), \gamma^* \tau(\varphi)$ if $\gamma \in \Gamma^-$ implies the existence of an exact sequence

$$1 \to \mathrm{Aut}(E, \varphi) \to \mathrm{Aut}_{\Gamma,\tau}(E, \varphi) \to \Gamma \to 1,$$

where $\mathrm{Aut}(E, \varphi)$ is the group of automorphisms of $(\mathrm{Aut}(E, \varphi)$ covering the identity and and $\mathrm{Aut}_{\Gamma,\tau}(E, \varphi)$ is the subgroup of $\mathrm{Aut}_{\Gamma,\tau}(E)$ defined by 3.2 defined by elements which send φ to φ if $\gamma \in \Gamma^+$ and φ to $\tau(\varphi)$ if $\gamma \in \Gamma^-$.

Since we are assuming that (E, φ) is simple $\mathrm{Aut}(E, \varphi) \cong Z' = Z \cap \ker \iota$ and hence we have an extension

$$1 \to Z' \to \mathrm{Aut}_{\Gamma,\tau}(E, \varphi) \to \Gamma \to 1.$$

This extension defines a cocycle $c \in Z^2_\tau(\Gamma, Z')$, and a c-twisted homomorphism $\Gamma \to \mathrm{Aut}_\Gamma(E, \varphi)$ with cocycle c, i.e., a (Γ, τ, c)-equivariant structure on (E, φ). It follows from (1) that the union should run over $[c] \in H^2_\tau(\Gamma, Z')$ and $[\sigma_i] \in R(\Gamma_{x_i}, H^{\mathbb{C}}/Z')$, where, recall that $[\sigma_i]$ is the projection of σ_i in $R(\Gamma_{x_i}, H^{\mathbb{C}}/Z')$. □

5. Equivariant structures and parabolic Higgs bundles

As in the previous section, let X be a compact Riemann surface, let $\Gamma \subset \mathrm{Aut}(X)$ be a finite subgroup. We will assume here that $\Gamma = \Gamma^+$. let $Y := X/\Gamma$ and $\pi_Y : X \to Y$ be the associated ramified covering map. The set of points $\mathscr{P} \subset X$ maps by π_Y to a set $\mathscr{S} \subset Y$. In this section we establish a correspondence between Γ-equivariant G-Higgs bundles over X and parabolic G-Higgs bundles over Y with parabolic points $\mathscr{S}$. This extends the well-known correspondences for vector bundles [26, 15, 28, 5, 2, 1], and principal bundles [38, 3]. In particular this implies that if $Z' = Z \cap \ker \iota$ and a G-Higgs bundle (E, φ) is equipped with a (Γ, c)-equivariant structure with $c \in Z^2(\Gamma, Z')$, then (E', φ) with $E' := E/Z'$ is a $G' = G/Z'$-Higgs bundle with a Γ-equivariant structure and hence is in correspondence with a parabolic G'-Higgs bundle over Y. It would be very interesting to give a parabolic description of the twisted equivariant structure on (E, φ).

5.1. **Parabolic G-Higgs bundles.** In this section Y is a compact Riemann surface, and G is a connected real reductive Lie group. We keep the same notation as in the previous sections for a maximal compact subgroup, isotropy representation, etc.

Let $T \subset H$ be a Cartan subgroup, and $\mathfrak{t}$ be its Lie algebra. We consider a Weyl alcove $\mathscr{A} \subset \mathfrak{t}$ (see [4]). Recall that if W is the Weyl group we have

$$\mathscr{A} \cong T/W \cong \mathrm{Conj}(H),$$

where $\mathrm{Conj}(H)$ is the set of conjugacy classes of H. Note that in contrast to the definition of alcove in [4], here $\mathscr{A}$ may contain some walls so that it is a fundamental domain for the action of the affine Weyl group.

Let $\mathscr{S} = \{y_1, \ldots, y_s\}$ be a finite set of distinct points of Y and $D = y_1 + \cdots + y_s$ be the corresponding effective divisor.

An element $\alpha \in \sqrt{-1}\mathscr{A}$ defines a parabolic subgroup of $P_\alpha \subset H^{\mathbb{C}}$ given by (2.3). Fix for every point $y_i \in \mathscr{S}$ an element $\alpha_i \in \sqrt{-1}\mathscr{A}$, and denote $\alpha = (\alpha_1, \cdots, \alpha_s)$.

A **parabolic G-Higgs bundle over $(Y, \mathscr{S})$ with weights** α is a pair (E, φ) consisting of a holomorphic $H^{\mathbb{C}}$-bundle E over Y equipped with a reduction of E_{y_i} to P_{α_i} and φ is a holomorphic section of $PE(\mathfrak{m}^{\mathbb{C}}) \otimes K(D))$, where $PE(\mathfrak{m}^{\mathbb{C}})$ is the sheaf of parabolic sections of $E(\mathfrak{m}^{\mathbb{C}})$ (see [4] for details). There are notions of stability, semistability and polystability similar to the ones we have already seen in previous sections ([4]).

To define a moduli space one has to fix for every point $y_i \in \mathscr{S}$ the projection $\mathscr{L}_i$ of the residue of φ in $\mathfrak{m}^0_{\alpha_i}/L_{\alpha_i}$, where $\mathfrak{m}^0_{\alpha_i}$ and L_{α_i} are defined as in Section 2.3. Denote $\mathscr{L} = (\mathscr{L}_1, \cdots, \mathscr{L}_s)$. We define $\mathcal{M}(Y, \mathscr{S}, G, \alpha, \mathscr{L})$ to be the **moduli space of parabolic G-Higgs bundles on $(Y, \mathscr{S})$ with weights $\alpha = (\alpha_1, \cdots, \alpha_s)$ and residues $\mathscr{L} = (\mathscr{L}_1, \cdots, \mathscr{L}_s)$.**

5.2. **Γ-equivariant Higgs bundles and parabolic Higgs bundles.** In this section we describe the correspondence between parabolic G-Higgs bundles on Y and Γ-equivariant G-Higgs bundles on X. For holomorphic vector bundles over a compact Riemann surface, this correspondence originated in [15] and was generalised to higher dimensions in [5]. The extension to Higgs vector bundles was carried out in [28], and for holomorphic principal bundles this correspondence is contained in [38] and [3].

First we begin with the data of a compact Riemann surface X and a finite subgroup $\Gamma \subset \mathrm{Aut}(X)$ consisting entirely of holomorphic automorphisms. Applying the smoothing process of [10, Sec. 2] to the orbifold X/Γ determines a compact Riemann surface Y and a holomorphic map $\pi_Y : X \to Y$ such that Γ is the group of deck transformations of the ramified cover π. Let $\{x_1, \ldots, x_r\}$ denote the ramification points of π and let $D = y_1 + \cdots + y_s$ denote the branch divisor. Each ramification point x_j has a non-trivial isotropy group denoted $\Gamma_{x_j} \subset \Gamma$ which is cyclic of order N_j. Let $N = |\Gamma|$ denote the order of the ramified cover $\pi_Y : X \to Y$.

Let $E \to X$ be a principal $H^{\mathbb{C}}$ bundle, and choose a lift of Γ to the group of C^∞ automorphisms of E. Via this lift, each isotropy group $\Gamma_{x_j} \cong \mathbb{Z}_{N_j}$ acts on the fibre E_{x_j} which determines a representation $\sigma_j \in R(\Gamma_{x_j}, H^{\mathbb{C}})$ (note that since we are considering equivariant rather than twisted equivariant bundles then the cocycle $c \in Z^2(\Gamma, Z')$ is trivial).

Let $\mathscr{C}_{x_j} \in \mathrm{Conj}(H)$ denote the conjugacy class of the generator γ_{x_j} of Γ_{x_j}, which is determined by the representation σ_j. Under the bijection between $\mathrm{Conj}(H)$ and a Weyl alcove $\mathscr{A}$ of H (see [4]) we thus have that each conjugacy class $\mathscr{C}_{x_j}$ corresponds to a weight $\alpha_j \in \sqrt{-1}\mathscr{A}$. Since $|\Gamma_{x_j}| = N_j$ then $e^{2\pi i N_j \alpha_j} = \mathrm{id} \in H^{\mathbb{C}}$. In the following we will always choose the weights α_j in the interior of the Weyl alcove $\sqrt{-1}\mathscr{A}$.

Given a branch point $y \in Y$ and two points $x, x' \in \pi^{-1}(y)$, there is a deck transformation $\gamma \in \Gamma$ such that $x' = \gamma \cdot x$, and the lift of γ to the group of automorphisms of E determines a map on the fibres $\gamma : E_x \to E_{x'}$. Moreover, the isotropy groups are conjugate $\Gamma_{x'} = \gamma \Gamma_x \gamma^{-1}$ and so the conjugacy classes $\mathscr{C}_x$ and $\mathscr{C}_{x'}$ are equal, and hence so are the weights in $\sqrt{-1}\mathscr{A}$ associated to these classes.

Now consider a Γ-equivariant Higgs structure on E, i.e. a holomorphic structure on E together with a Higgs field φ such that (E, φ) is preserved by the action of Γ. For each ramification point x_j, choose a small neighbourhood U_j such that the bundle is trivial $E|_{U_j} \cong U_j \times H^{\mathbb{C}}$ and the Γ-action is trivial

$$e^{\frac{2\pi i}{N_j}} \cdot (z, g) = (e^{\frac{2\pi i}{N_j}} z, e^{2\pi i \alpha_j} \cdot g) \tag{5.1}$$

(as explained in [38], the existence of this trivialisation follows from the equivariant Oka principle of [19]). We now show that after gauging by $z^{-N_j \alpha_j}$ on each trivialisation for $j = 1, \ldots, r$ then the Higgs pair (E, φ) descends to a parabolic Higgs bundle on the quotient $(X \setminus \mathscr{P})/\Gamma$, where the weight at the branch point $\pi(x_j)$ is α_j. This is known for holomorphic vector bundles (cf. [15], [5]) and holomorphic principal bundles (cf. [38], [3]), and so to describe the correspondence for Higgs bundles it only remains to describe the residue of the Higgs field at each branch point in Y, which is a local computation on each neighbourhood U_j. This was worked out for Higgs vector bundles in [28], however this has not appeared in the literature for general G-Higgs bundles and so we include the details below.

Locally, the Higgs field on E has the form $\varphi(z) = f(z)dz$, where $f(z) : U_j \to \mathfrak{m}^{\mathbb{C}}$ is holomorphic. The action of $\mathrm{Ad}_{e^{2\pi i \alpha_j}}$ decomposes $\mathfrak{m}^{\mathbb{C}}$ into eigenspaces

$$\mathfrak{m}^{\mathbb{C}} = \bigoplus_{\beta} \mathfrak{m}^{\mathbb{C}}_{\beta}$$

where $\mathfrak{m}^{\mathbb{C}}_{\beta}$ denotes the eigenspace with eigenvalue $e^{2\pi i \beta}$. Note that each $N_j \beta$ is an integer since $e^{2\pi i N_j \alpha_j} = \mathrm{id}$, and since α_j is in the interior of the Weyl alcove then each eigenvalue is strictly less than one. Let $f = \bigoplus_{\beta} f_{\beta}$ be the corresponding decomposition of f. Since each f_{β} is holomorphic then we can write it as a power series

$$f_{\beta}(z) = \sum_{k=0}^{\infty} a_k^{\beta} z^k$$

with a_k^{β} taking values in $\mathfrak{m}^{\mathbb{C}}_{\beta}$. The induced action of $e^{\frac{2\pi i}{N_j}}$ on φ is given by

$$e^{\frac{2\pi i}{N_j}} \cdot \varphi(z) = \mathrm{Ad}_{e^{2\pi i \alpha_j}} \left(f \left(e^{\frac{2\pi i}{N_j}} z \right) \right) e^{\frac{2\pi i}{N_j}} dz.$$

Therefore, the action on the component $\varphi_\beta = f_\beta dz$ is

$$e^{\frac{2\pi i}{N_j}} \cdot f_\beta(z) dz = e^{2\pi i \beta} \sum_{k=0}^{\infty} a_k^\beta e^{\frac{2\pi i k}{N_j}} z^k \, e^{\frac{2\pi i}{N_j}} dz = \sum_{k=0}^{\infty} a_k^\beta e^{\frac{2\pi i (k+1)}{N_j}} e^{2\pi i \beta} z^k \, dz.$$

If φ is invariant under the action of $\mathbb{Z}_{N_j} \cong \Gamma_{x_j}$ then we see that $a_k^\beta \neq 0$ implies that $k = N_j \ell - N_j \beta - 1$ for some $\ell \in \mathbb{Z}$. Therefore

$$f_\beta(z) dz = \begin{cases} z^{-N_j \beta} \sum_{\ell=0}^{\infty} a_{N_j \ell - N_j \beta - 1}^\beta z^{N_j \ell} \, z^{-1} dz & \text{if } \beta < 0 \\ z^{-N_j \beta} \sum_{\ell=1}^{\infty} a_{N_j \ell - N_j \beta - 1}^\beta z^{N_j \ell} \, z^{-1} dz & \text{if } 0 \leqslant \beta < 1 \end{cases}$$

where the two distinct cases come from the requirement that f_β is holomorphic and hence the power series has non-negative powers of z. To simplify the notation, we will use $b_\ell^\beta = a_{N_j \ell - N_j \beta - 1}^\beta$ in the sequel. On the punctured disk $U_j \smallsetminus \{0\}$, apply the meromorphic gauge transformation $g(z) = z^{N_j \alpha_j} = e^{N_j \alpha_j \log z}$ (note that this is well-defined on the punctured neighbourhood $U_j \smallsetminus \{x_j\}$ since $e^{2\pi i N_j \alpha_j} = \mathrm{id}$). We have $g(z) \cdot \varphi(z) = \sum_\beta g(z) \cdot f_\beta(z) dz$ where

$$g(z) \cdot f_\beta(z) dz = \begin{cases} \sum_{\ell=0}^{\infty} b_\ell^\beta z^{N_j \ell} \, z^{-1} dz & \text{if } \beta < 0 \\ \sum_{\ell=0}^{\infty} b_{\ell+1}^\beta z^{N_j \ell} \, z^{N_j - 1} dz & \text{if } 0 \leqslant \beta < 1 \end{cases}$$

Therefore, after applying the meromorphic gauge transformation $g(z)$, the residue of $g(z) \cdot f_\beta(z)$ is zero if $\beta \geqslant 0$ and equal to b_0^β if $\beta < 0$. Now let $V = \pi(U_j) \subset Y$ and note that (5.1) implies that $\pi : U_j \to V$ is given by $z \mapsto z^{N_j}$. Then $w = z^{N_j}$ satisfies $w^{-1} dw = N_j z^{-1} dz$ and so $g(z) \cdot f_\beta(z)$ can be written as a function of w, i.e. it descends to the quotient $(U_j \smallsetminus \{x_j\}) / \Gamma_{x_j}$

$$g(z) \cdot f_\beta(z) = f'_\beta(w) = \begin{cases} \sum_{\ell=0}^{\infty} b_\ell^\beta w^\ell \, \frac{1}{N_j} w^{-1} dw & \text{if } \beta < 0 \\ \sum_{\ell=0}^{\infty} b_{\ell+1}^\beta w^\ell \, \frac{1}{N_j} dw & \text{if } 0 \leqslant \beta < 1 \end{cases}$$

Therefore the Γ-invariant Higgs bundle (E, φ) on X defines a parabolic Higgs bundle (E', φ') on Y with Higgs field $\varphi' \in \Gamma\left(PE'(\mathfrak{m}^{\mathbb{C}}) \otimes K(D)\right)$. In particular, the residue of the Higgs field $\varphi'(w) = f'(w) dw$ is $\bigoplus_{\beta < 0} b_0^\beta$ which is nilpotent and so the projection to $\mathfrak{m}_{\alpha_i}^0 / L_{\alpha_j}$ is zero.

Therefore the Γ-equivariant Higgs bundle (E, φ) on X with isotropy representations σ corresponding to weights $\alpha_j \in \sqrt{-1}\mathcal{A}$ in the interior of the Weyl alcove determines a parabolic Higgs bundle (E', φ') on Y with parabolic points $\{y_1, \ldots, y_s\} = \pi(\{x_1, \ldots, x_r\})$, conjugacy classes $\mathscr{C}'_{\pi(x_j)} = \mathscr{C}_{x_j}$ determined by α_j and a parabolic Higgs field with nilpotent residues. Moreover, gauge equivalent Γ-equivariant Higgs bundles on X descend to parabolic gauge-equivalent parabolic Higgs bundles on Y.

Conversely, given a parabolic G-Higgs bundle (E', φ') on Y with nilpotent residues at each parabolic point, as above let V be a neighbourhood of a branch point y such that the bundle is trivial over $V \smallsetminus \{y\}$ with weight $\alpha_j \in \sqrt{-1}\mathcal{A}$ such that $e^{2\pi i N_j \alpha_j} = \mathrm{id}$. Since the residues are nilpotent then the Higgs field $\varphi' \in \Gamma\left(PE'(\mathfrak{m}^{\mathbb{C}}) \otimes K(D)\right)$ has the form

$$f'_\beta(w) = \begin{cases} \sum_{\ell=0}^{\infty} c_\ell^\beta w^\ell \, w^{-1} dw & \text{if } \beta < 0 \\ \sum_{\ell=0}^{\infty} c_\ell^\beta w^\ell \, dw & \text{if } 0 \leqslant \beta < 1 \end{cases}$$

After pulling back by the ramified covering map $z \mapsto z^{N_j} = w$, the Higgs field $\varphi(z) = f(z)dz$ upstairs has the form

$$f_\beta(z) = \begin{cases} \sum_{\ell=0}^{\infty} c_\ell^\beta z^{N_j \ell} \, N_j z^{-1} dz & \text{if } \beta < 0 \\ \sum_{\ell=0}^{\infty} c_\ell^\beta z^{N_j \ell} \, N_j z^{N_j - 1} dz & \text{if } 0 \leqslant \beta < 1 \end{cases}$$

Applying the gauge transformation $g(z) = z^{-N_j \alpha_j}$ (once again, $e^{2\pi i N_j \alpha_j} = \mathrm{id}$ implies that this is well-defined on the punctured neighbourhood $U_j \setminus \{x_j\}$) gives us

$$g(z) \cdot f_\beta(z) = \begin{cases} z^{-N_j \beta} \sum_{\ell=0}^{\infty} c_\ell^\beta z^{N_j \ell} \, N_j z^{-1} dz & \text{if } \beta < 0 \\ z^{-N_j \beta} \sum_{\ell=0}^{\infty} c_\ell^\beta z^{N_j \ell} \, N_j z^{N_j - 1} dz & \text{if } 0 \leqslant \beta < 1 \end{cases}$$

and the same argument as before shows that this is holomorphic and invariant under the action of $\mathbb{Z}_{N_j}$ determined by $\alpha_j \in \sqrt{-1}\mathcal{A}$. Therefore the parabolic Higgs bundle on Y determines a Γ-equivariant Higgs bundle on X.

Now that we have established the correspondence, it only remains to show that the notions of stability, semistability and polystability are also in correspondence. In the case of holomorphic principal bundles, the results of [38, Sec. 2.2] show that, via the correspondence described above, a stable Γ-equivariant bundle upstairs on X corresponds to a stable parabolic bundle on Y. Moreover, the degree of any parabolic reduction of structure group on $E \to X$ is related to the parabolic degree of a parabolic reduction of structure group on $E' \to Y$ by a factor of $\frac{1}{|\Gamma|}$.

For Higgs bundles, the only modification is to restrict to reductions of structure group which are compatible with the Higgs field as described in [4, Sec. 3.2]. For the Higgs bundle (E, φ) over X, given $s \in \sqrt{-1}\mathfrak{h}$ and a Γ-invariant holomorphic reduction $\eta \in \Omega^0(E(H^{\mathbb{C}}/P_s))$ such that $\varphi \in H^0(X, E_\eta(\mathfrak{m}_s) \otimes K)$, the Γ-invariance of the Higgs field φ implies that the induced reduction of structure group on the parabolic bundle (E', φ') over $Y \setminus \mathscr{S}$ is compatible with the Higgs field, i.e. $\varphi'|_{Y \setminus \mathscr{S}} \in H^0(Y \setminus \mathscr{S}, E'_\eta(\mathfrak{m}_s) \otimes K)$. Conversely, a reduction of structure group on the parabolic bundle (E', φ') over $Y \setminus \mathscr{S}$ which is compatible with the Higgs field φ' lifts to a reduction of (E, φ) over X compatible with φ. Since the degree on X is related to the parabolic degree on Y by a factor of $\frac{1}{|\Gamma|}$ (cf. [38, Sec. 2.3]) then the notion of Γ-equivariant Higgs stability (resp. semistability and polystability) upstairs on X corresponds to the notion of parabolic Higgs stability (resp. semistability and polystability) downstairs on Y.

Therefore we have proved the following bijection of moduli spaces.

Theorem 5.1. *The correspondence described above defines a bijection*

$$\mathcal{M}(X, G, \Gamma, \mathrm{id}, \sigma) \to \mathcal{M}(Y, \mathscr{S}, G, \alpha, 0).$$

5.3. **τ-Twisted Γ-equivariant structures and pseudoreal parabolic Higgs bundles.** In this section we assume that Γ contains antiholomorphic automorphisms Γ contains antiholomorphic automorphisms of X, that is, Γ is given by an extension

$$1 \to \Gamma^+ \to \Gamma \to \mathbb{Z}/2 \to 1$$

defined by (2.2). We also assume that the (Γ, τ, c)-equivariant structures on the G-Higgs bundles over X are such that the restriction of the cocycle c to Γ^+ is trivial. In this situation c defines a cocycle $\widetilde{c} \in Z^2_\tau(\mathbb{Z}/2, Z')$ where the action of $\mathbb{Z}/2 = \Gamma/\Gamma^+$ is the one induced by (3.1). The cocycle $\widetilde{c}$ defines a pseudoreal structure on the parabolic G-Higgs bundle on $Y := X/\Gamma^+$ constructed in the previous section. Pseudoreal structures of parabolic G-Higgs bundles are studied in [11], generalising the theory of pseudoreal G-Higgs bundles is well-understood [8, 7, 6]. One thus has a correspondence similar to the one in Theorem 5.1 also in this situation.

6. Twisted equivariant structures and representations

In this section, S is an oriented smooth compact surface of genus $g \geqslant 2$, X is a Riemann surface, whose underlying smooth surface is S. The Lie group G is a real form of a complex semisimple Lie group $G^{\mathbb{C}}$, and τ is a conjugation of $G^{\mathbb{C}}$ defining a compact real form of $G^{\mathbb{C}}$, and preserving G. Finally, Γ is a subgroup of $\mathrm{Aut}(X)$.

6.1. **Twisted equivariant Higgs bundles and the orbifold fundamental group.** Exploiting Proposition 2.7 and Theorem 4.4, we will give an interpretation of the fix-point locus $\mathcal{R}(S, G)^\Gamma$ in terms of representations of the Γ-orbifold fundamental group $\pi_1(S, \Gamma)$ of S (see [8], for example, for a definition). This group fits into a short exact sequence

$$1 \to \pi_1(S) \to \pi_1(S, \Gamma) \to \Gamma \to 1.$$

Let $c \in Z^2_\tau(\Gamma, Z)$ be a 2-cocycle, where Z is the centre of G. Recall that Γ acts on G as $g^\gamma = g$ if $\gamma \in \Gamma^+$ and $g^\gamma = \tau(g)$ if $\gamma \in \Gamma^-$, for $g \in G$, inducing an action on Z. Let also

$$\tau^\gamma = \begin{cases} \mathrm{Id} & \text{if } \ \gamma \in \Gamma^+ \\ \tau & \text{if } \ \gamma \in \Gamma^-. \end{cases}$$

We consider a group $\widehat{G} = \widehat{G}(\Gamma, \tau, c)$, whose set is $G \times \Gamma$, and the group structure is defined by

$$(g_1, \gamma_1) \cdot (g_2, \gamma_2) = (g_1 \tau^{\gamma_1}(g_2) c(\gamma_1, \gamma_2), \gamma_1\gamma_2),$$

for $g_1, g_2 \in G$ and $\gamma_1, \gamma_2 \in \Gamma$. Define

$$\mathcal{R}(S, G, \Gamma, \tau, c) \subset \mathcal{R}(S, G)$$

to be the subset corresponding to those ρ which extend to a homomorphism $\widehat{\rho} : \pi_1(S, \Gamma) \to \widehat{G}$ making the following diagram commutative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_1(S) & \longrightarrow & \pi_1(S,\Gamma) & \longrightarrow & \Gamma & \longrightarrow & 1 \\
 & & {\scriptstyle\rho}\big\downarrow & & {\scriptstyle\widehat{\rho}}\big\downarrow & & {\scriptstyle\mathrm{Id}}\big\downarrow & & \\
1 & \longrightarrow & G & \longrightarrow & \widehat{G} & \longrightarrow & \Gamma & \longrightarrow & 1.
\end{array}$$

Following the arguments in [8] (see also [7, 6]), we can extend the non-abelian Hodge correspondence to the twisted equivariant case and prove the following.

Theorem 6.1. *Under the non-abelian Hodge correspondence given by Theorem 2.5 one has the homeomorphism*

$$\mathcal{R}(S, G, \Gamma, \tau, c) \xrightarrow{\cong} \mathcal{M}(X, G, \Gamma, \tau, c).$$

Remark 6.2. Here, we are identifying $\mathcal{M}(X, G, \Gamma, \tau, c)$ with its image in $\mathcal{M}(X, G)$.

Remark 6.3. Fixing the elements σ_i at the points $x_i \in \mathscr{P}$ to define the moduli space $\mathcal{M}(X, G, \Gamma, \tau, c, \sigma)$ with $\sigma = (\sigma_1, \cdots, \sigma_r)$, corresponds to fixing a cyclic element for the image under the representation of a loop around the point x_i. The moduli space corresponding to $\mathcal{M}(X, G, \Gamma, \tau, c, \sigma)$ will be denoted by $\mathcal{R}(S, G, \Gamma, \tau, c, \sigma)$.

As a corollary of Theorems 4.4 and 6.1 we have the following.

Theorem 6.4. *(1) For any cocycle $c \in Z^2_\tau(\Gamma, Z)$*

$$\mathcal{R}(S, G, \Gamma, \tau, c) \subset \mathcal{R}(S, G)^\Gamma.$$

(2) Let $\mathcal{R}_*(S, G) \subset \mathcal{R}(S, G)$ be the subvariety of irreducible representations, then

$$\mathcal{R}_*(S, G)^\Gamma \subset \bigcup_{[c] \in H^2_\tau(\Gamma, Z)} \mathcal{R}(S, G, \Gamma, \tau, c).$$

6.2. **The orbifold fundamental group and punctured surfaces.** Assume now, as in Section 5.2, that $\Gamma = \Gamma^+$ and that the cocycle c is trivial. Let $\mathscr{S}$ be the set of points in S/Γ corresponding to the points $\mathscr{P} \subset S$ (see Section 5.2). Then, combining Theorems 5.1 and 6.1 with the non-abelian Hodge correspondence for punctured surfaces, proved in [4], we have the following.

Theorem 6.5. *There is a bijection between $\mathcal{R}(S, G, \Gamma, \tau, c = 1, \sigma)$ and $\mathcal{R}(S/\Gamma \smallsetminus \mathscr{S}, G)$ with conjugacy classes around the points in $\mathscr{S}$ determined by σ.*

We now assume, as in Section 5.3, that Γ contains antiholomorphic automorphisms and that the restriction of the cocycle c to Γ^+ is trivial. As mentioned in Section 5.3, in this situation c defines a cocycle $\widetilde{c} \in Z^2_\tau(\mathbb{Z}/2, Z)$ where the action of $\mathbb{Z}/2 = \Gamma/\Gamma^+$ is the one induced by (3.1). We can then consider the $\mathbb{Z}/2$-orbifold fundamental group $\pi_1(S/\Gamma^+ \smallsetminus \mathscr{S}, \mathbb{Z}/2)$ for the residual action of $\mathbb{Z}/2 = \Gamma/\Gamma^+$ on S/Γ^+, which fits in a short exact sequence

$$1 \to \pi_1(S/\Gamma^+ \smallsetminus \mathscr{S}) \to \pi_1(S/\Gamma^+ \smallsetminus \mathscr{S}, \mathbb{Z}/2) \to \mathbb{Z}/2 \to 1.$$

Here $\mathscr{S}$ is the set of points S/Γ^+ corresponding to the set $\mathscr{P} \subset S$.

We define the group $\widehat{G} = \widehat{G}(\tau, c)$, whose set is $G \times \mathbb{Z}/2$, and the group structure is defined by

$$(g_1, e_1) \cdot (g_2, e_2) = (g_1 \tau^{e_1}(g_2) \widetilde{c}(e_1, e_2), e_1 e_2),$$

for $g_1, g_2 \in G$ and $e_1, e_2 \in \mathbb{Z}/2$. Here $\tau^{e_1} = \mathrm{Id}$ if $e_1 = 1$ and $\tau^{e_1} = \tau$ if $e_1 = -1$. Define

$$\mathcal{R}(S/\Gamma^+ \smallsetminus \mathscr{S}, G, \tau, c) \subset \mathcal{R}(S/\Gamma^+ \smallsetminus \mathscr{S}, G)$$

corresponding to those ρ which extend to a homomorphism $\widehat{\rho}: \pi_1(S/\Gamma^+ \smallsetminus \mathscr{S}, \mathbb{Z}/2) \to \widehat{G}$ making the following diagram commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(S/\Gamma^+ \smallsetminus \mathscr{S}) & \longrightarrow & \pi_1(S/\Gamma^+ \smallsetminus \mathscr{S}, \mathbb{Z}/2) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 1 \\ & & {\scriptstyle\rho}\downarrow & & {\scriptstyle\widehat{\rho}}\downarrow & & {\scriptstyle\mathrm{Id}}\downarrow & & \\ 1 & \longrightarrow & G & \longrightarrow & \widehat{G} & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 1. \end{array}$$

From the discussion in Section 5.3 we conclude the following.

Theorem 6.6. *There is a bijection between $\mathcal{R}(S, G, \Gamma, \tau, c, \sigma)$ and $\mathcal{R}(S/\Gamma^+ \smallsetminus \mathscr{S}, G, \tau, c)$ with conjugacy classes around the points in $\mathscr{S}$ determined by σ.*

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Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM, Nicolás Cabrera, 13–15, 28049 Madrid, Spain

E-mail address: oscar.garcia-prada@icmat.es

Department of Mathematics, James College, Campus West, University of York, YO10 5DD, United Kingdom

E-mail address: graeme.wilkin@york.ac.uk