# ANTI-HOLOMORPHIC INVOLUTIVE ISOMETRY OF HYPER-KÄHLER MANIFOLDS AND BRANES

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Abstract. We study complex Lagrangian submanifolds of a compact hyper-Kähler manifold and prove two results: (a) that an involution of a hyper-Kähler manifold which is antiholomorphic with respect to one complex structure and which acts non-trivially on the corresponding symplectic form always has a fixed point locus which is complex Lagrangian with respect to one of the other complex structures, and (b) there exist Lagrangian submanifolds which are complex with respect to one complex structure and are not the fixed point locus of any involution which is anti-holomorphic with respect to one of the other complex structures.

## 1. Introduction

Let $(X, \omega_I, \omega_J, \omega_K, g)$ be a hyper-Kähler manifold. In this paper we study submanifolds of $X$ which are complex Lagrangian (of $B$ type) with respect to one of the complex structures $I, J, K$ and Lagrangian (of $A$ type) with respect to the Kähler form associated to the other two complex structures. The motivation for this comes from the paper of Kapustin and Witten [11], where they study such submanifolds of the moduli space of Higgs bundles $\mathcal{M}_H$ over a compact Riemann surface. The most interesting examples are called $(B, A, A)$ branes, $(A, B, A)$ branes and $(A, A, B)$ branes (cf. [11, Sec. 5.6]). Recently, a number of authors have constructed discrete families of $(A, B, A)$ branes in $\mathcal{M}_H$ via anti-holomorphic involutions on $\mathcal{M}_H$ (see [1], [2], [4], [9]).

In this paper we prove two results related to complex Lagrangian submanifolds of hyper-Kähler manifolds. The first one (see Section 2) describes the geometric structure on the fixed point locus of an anti-holomorphic involution.

**Theorem 1.1.** *Let $X$ be a hyper-Kähler manifold of complex dimension $2d$ and let $I$ be one of the complex structures with associated Kähler form $\omega_I$. Let $\sigma : X \longrightarrow X$ be an involution such that $\sigma$ is anti-holomorphic with respect to $I$ and that $\sigma^*\omega_I = -\omega_I$. Fix an element $\theta \in H^0(X, \Omega_X^2) \setminus \{0\}$ (holomorphic with respect to $I$) such that $\sigma^*\theta = \bar{\theta}$ and the pointwise norm of $\theta$ is $2\sqrt{d}$. Let $\theta_J$ and $\theta_K$ be the holomorphic symplectic forms with respect to $J$ and $K$ respectively. Assume that the fixed point locus $S = X^\sigma \subset X$ is nonempty. Then*

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- *the fixed point locus $S$ is a special Lagrangian submanifold with respect to both $(\omega_I, \theta^d)$ and $(\omega_K, \theta_K^d)$, and*
- *$S$ is a complex Lagrangian manifold with respect to $(J, \theta_J)$.*

The second result (see Section 3) is that deformations of these examples give complex Lagrangian submanifolds that are not fixed points of any anti-holomorphic involution.

**Theorem 1.2.** *Let $X$ be a hyper-Kähler manifold of complex dimension $2d$ and let $I$ be one of the complex structures with associated Kähler form $\omega_I$. Let $\sigma : X \longrightarrow X$ be an involution such that $\sigma$ is anti-holomorphic with respect to $I$ and that $\sigma^*\omega_I = -\omega_I$. Suppose that the fixed point set $S$ is compact and has positive first Betti number. Then there exist continuous families of complex Lagrangian submanifolds of $X$ that are not fixed points of an anti-holomorphic involution $\sigma : X \longrightarrow X$.*

It is natural to ask whether one can construct new examples of $(A, B, A)$ branes in the moduli space of Higgs bundles via deformations of the discrete families in [1], [2], [4] and [9].

## 2. The fixed point set of an anti-holomorphic involution on a hyper-Kähler manifold

In this section we prove Theorem 1.1, which describe the geometric structures on the fixed point locus of the involution $\sigma$.

A hyper-Kähler manifold is a quintuple $(X\,, g\,, I\,, J\,, K)$, where $X$ is a connected smooth manifold, $g$ is a Riemannian metric on $X$ and $I$, $J$, $K$ are integrable almost complex structures on $X$ such that

(1) $I$, $J$ and $K$ are orthogonal with respect to $g$,
(2) the Hermitian forms for $(I\,, g)$, $(J\,, g)$ and $(K\,, g)$ are closed, and
(3) $K = IJ = -JI$, $J = -IK = KI$ and $I = JK = -KJ$.

(See [10].) The hyper-Kähler manifold is called *irreducible* if the holonomy of the Levi–Civita connection corresponding to $g$ has holonomy $\mathrm{Sp}(m/4)$, where $m = \dim_{\mathbb{R}} X$.

Let $(X\,, g\,, I\,, J\,, K)$ be a compact irreducible hyper-Kähler manifold of real dimension $4d$. Let $\omega$ be the Kähler form for $(I\,, g)$. Let

$$\sigma : X \longrightarrow X \tag{2.1}$$

be an anti-holomorphic involution with respect to $I$ such that

$$\sigma^*\omega = -\omega\,.$$

Since $\sigma$ is anti-holomorphic with respect to $I$, we have $\sigma^* I = -I$. We should emphasize that a general hyper-Kähler manifold does not admit such an involution. See [12], [5] for examples of $K3$ surfaces admitting such involution (see also [14], [15]).

Henceforth, unless specified otherwise, $I$ would be taken as the complex structure on $X$. So all holomorphic objects on $X$ are with respect to $I$ (unless specified otherwise). For example, $\Omega_X$ denotes the holomorphic cotangent bundle of $X$ with respect to $I$.

Since the hyper-Kähler manifold $X$ is irreducible, we have $\dim H^0(X,\, \Omega^2_X) \,=\, 1$ [3, p. 762, Proposition 3(ii)]. Take any nonzero element $\theta' \,\in\, H^0(X,\, \Omega^2_X)$. We have

$$\sigma^*\theta' \,=\, c\cdot\overline{\theta'}\,, \tag{2.2}$$

where $c \,\in\, \mathbb{C}\setminus\{0\}$. From (2.2) we have

$$\sigma^*\sigma^*\theta' \,=\, c\bar{c}\theta'\,.$$

Since $\sigma\circ\sigma \,=\, \mathrm{Id}_X$, this implies that $|c| \,=\, 1$. Setting $\theta \,:=\, \sqrt{\bar{c}}\,\theta'$, we obtain $\sigma^*\theta \,=\, \overline{\theta}$.

Fix an element $\theta \,\in\, H^0(X,\, \Omega^2_X)\setminus\{0\}$ such that

(1) $\sigma^*\theta \,=\, \overline{\theta}$, and
(2) the pointwise norm of $\theta$ with respect to $g$ is $2\sqrt{d}$.

Such a section $\theta$ exists because the holomorphic 2-forms on $X$ are covariant constant with respect to the Levi-Civita connection on $(X\,,g)$. Note that any two such choices differ by multiplication with $-1$.

**Lemma 2.1.** *The involution $\sigma$ has the following properties:*

(1) *It is an isometry for the Riemannian structure $g$ on $M$.*
(2) $\sigma^*\mathrm{Re}(\theta) \,=\, \mathrm{Re}(\theta)$, *where* $\mathrm{Re}(\theta)$ *is the real part of* $\theta$.
(3) $\sigma^*\mathrm{Im}(\theta) \,=\, -\mathrm{Im}(\theta)$, *where* $\mathrm{Im}(\theta)$ *is the imaginary part of* $\theta$.

*Proof.* We recall that $\omega(\alpha\,,\beta) \,=\, g(I(\alpha)\,,\beta)$, where $\alpha$ and $\beta$ are real tangent vectors at a point of $X$. Since $\sigma^*I \,=\, -I$ and $\sigma^*\omega \,=\, -\omega$, the first statement follows. The remaining two statements follow from the fact that $\sigma^*\theta \,=\, \overline{\theta}$. □

Let

$$S \,=\, X^\sigma \,\subset\, X \tag{2.3}$$

be the subset fixed pointwise by $\sigma$. Assume that $S$ is nonempty. This $S$ is a real manifold of dimension $2d$, but it need not be connected. The complex dimension of $X$ being even, the involution $\sigma$ is orientation preserving. Let $N$ be the normal bundle of $S$. As the differential of $\sigma$ acts on $N$ as multiplication by $-1$, and the rank of $N$ is even, we conclude that $d\sigma$ preserves the orientation of $N$. Since $\sigma$ is orientation preserving and $d\sigma$ preserves the orientation of $N$, it follows that $S$ is oriented.

**Lemma 2.2.** *The fixed point locus $S$ is a special Lagrangian submanifold with respect to $(\omega\,,\theta^d)$.*

*Proof.* Let

$$\iota\,:\, S\,\hookrightarrow\, X$$

be the inclusion map. We have $\iota^*\sigma^*\omega = \iota^*\omega$ because $S$ is fixed pointwise by $\sigma$. On the other hand, $\sigma^*\omega = -\omega$. Combining these we get that

$$\iota^*\omega = 0 .$$

Therefore, $S$ is Lagrangian with respect to the Kähler form $\omega$.

To prove that $S$ is special Lagrangian we need to show that

$$\iota^*\mathrm{Im}(\theta^d) = 0 , \tag{2.4}$$

where $\mathrm{Im}(\theta^d)$ is the imaginary part of the $(2d\,,0)$-form $\theta^d$.

We have $\iota^*\sigma^*\theta = \iota^*\theta$ because $S$ is fixed pointwise by $\sigma$. On the other hand, from Lemma 2.1(3),

$$\iota^*\mathrm{Im}(\theta) = -\iota^*\mathrm{Im}(\theta) .$$

Combining these it follows that $\iota^*\mathrm{Im}(\theta) = 0$. This immediately implies that (2.4) holds. So $S$ is a special Lagrangian manifold. □

Let $J$ be the almost complex structure on $X$ uniquely given by the equation

$$\mathrm{Re}(\theta)(\alpha\,,\beta) = g(J(\alpha)\,,\beta) , \tag{2.5}$$

where $\alpha$ and $\beta$ are real tangent vectors at a point of $X$. Similarly, $K$ is the almost complex structure on $X$ that satisfies the equation

$$\mathrm{Im}(\theta)(\alpha\,,\beta) = g(K(\alpha)\,,\beta) . \tag{2.6}$$

Both $(X\,,J\,,g)$ and $(X\,,K\,,g)$ are Kähler manifolds, in particular, both $J$ and $K$ are integrable. Let $\omega_J$ and $\omega_K$ be the Kähler forms for $(J\,,g)$ and $(K\,,g)$ respectively. So, from (2.5) and (2.6),

$$\mathrm{Re}(\theta)(\alpha\,,\beta) = \omega_J(\alpha\,,\beta) \quad \text{and} \quad \mathrm{Im}(\theta)(\alpha\,,\beta) = \omega_K(\alpha\,,\beta) \tag{2.7}$$

We recall that

$$\theta_J := \omega_K + \sqrt{-1}\omega \quad \text{and} \quad \theta_K := \omega_J + \sqrt{-1}\omega \tag{2.8}$$

are holomorphic symplectic forms on the Kähler manifolds $(X\,,J\,,g)$ and $(X\,,K\,,g)$ respectively.

**Proposition 2.3.** *The involution $\sigma$ is holomorphic with respect to $J$, and it is anti-holomorphic with respect to $K$. Also,*

$$\sigma^*\omega_J = \omega_J \quad \textit{and} \quad \sigma^*\omega_K = -\omega_K .$$

*Furthermore,*

$$\sigma^*\theta_J = -\theta_J \quad \textit{and} \quad \sigma^*\theta_K = \overline{\theta_K} ,$$

*where $\theta_J$ and $\theta_K$ are defined in* (2.8).

*Proof.* In view of Lemma 2.1(1) and Lemma 2.1(2), from (2.5) we conclude that $\sigma$ preserves $J$. Similarly, from Lemma 2.1(1), Lemma 2.1(2) and (2.6) it follows that $\sigma$ takes $K$ to $-K$.

From Lemma 2.1(2) and (2.7) it follows that $\sigma^*\omega_J = \omega_J$. From 2.1(3) and (2.7) it follows that $\sigma^*\omega_K = -\omega_K$.

Since $\sigma^*\omega = -\omega$ and $\sigma^*\omega_K = -\omega_K$, from (2.8) it follows that $\sigma^*\theta_J = -\theta_J$. Also, as $\sigma^*\omega_J = \omega_J$, we have $\sigma^*\theta_K = \overline{\theta_K}$. □

**Corollary 2.4.** *The fixed point locus $S$ is a special Lagrangian submanifold with respect to $(\omega_K, (\theta_K)^d)$.*

*Proof.* Since $\sigma$ is anti-holomorphic with respect to $K$, $\sigma^*\omega_K = -\omega_K$ and $\sigma^*\theta_K = \overline{\theta_K}$, the proof of Lemma 2.2 goes through. □

**Corollary 2.5.** *The fixed point locus $S$ is a complex Lagrangian submanifold with respect to $(J, \theta_J)$.*

*Proof.* Since $\sigma$ is holomorphic with respect to $J$, it follows that $S$ is a complex submanifold with respect to $J$. As $\sigma^*\theta_J = -\theta_J$, it is straightforward to deduce that $S$ is a Lagrangian submanifold with respect to the holomorphic symplectic structure $\theta_J$. □

## 3. Deformations of complex Lagrangian submanifolds

Let $(X, \omega^c)$ be a complex manifold of complex dimension $2d$ equipped with a holomorphic symplectic form $\omega^c$. Let $Y \subset X$ be a complex Lagrangian submanifold (i.e., $\omega^c$ vanishes on $Y$). Suppose also that $X$ has a Kähler structure and that $Y$ is compact. In [8], Hitchin studies the deformation space of compact complex Lagrangian submanifolds near $Y$ and shows that

(1) the deformations are unobstructed (see also [13]),
(2) there exists a local moduli space $M$ with real dimension equal to the first Betti number of $Y$, $\dim_{\mathbb{R}} T_{[Y]}M = b_1(Y)$, and
(3) $M$ has a naturally induced special Kähler structure.

For an irreducible compact hyper-Kähler manifold of complex dimension $2d$, we have $TX = \Omega_X^{2d-1}$, where $TX$ is the holomorphic tangent bundle, because $\Omega_X^{2d}$ is holomorphically trivial. Hence

$$H^0(X, TX) = H^0(X, \Omega_X^{2d-1}) = 0$$

(see [3, p. 762, Proposition 3(ii)]). This implies that the group of holomorphic automorphisms of $X$ is discrete. Any two anti-holomorphic involutions on $X$ differ by a holomorphic automorphism, so there are at most countably many anti-holomorphic involutions $X$.

Therefore we have the following:

**Theorem 3.1.** *Let $(X, \omega)$ be a compact hyper-Kähler manifold equipped with an anti-holomorphic involution $\sigma$ such that $\sigma^*\omega = -\omega$. Suppose that the fixed point set $S$ satisfies $b_1(S) > 0$. Then there exist complex Lagrangian submanifolds of $X$ that are not fixed points of an anti-holomorphic involution $X \longrightarrow X$.*

Let $X$ be a $K3$ surface with an involution $\sigma$ which is anti-holomorphic with respect to one of the complex structures $I$. Then there is a $\sigma$-invariant hyper-Kähler metric on $X$ and $\sigma$ is anti-symplectic with respect to $\omega_I$ and holomorphic with respect to one of the complex structures orthogonal to the original one (cf. [6, pp. 21-22]).

Explicit examples of such anti-holomorphic involutions of $K3$ surfaces with $b_1(S) > 0$ can be found in [12] (see [12, Section 3.4] and [12, Section 3.8]). Gross and Wilson have also studied such involutions on $K3$ surfaces in the context of mirror symmetry in [7].

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