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HITCHIN'S EQUATIONS ON A NONORIENTABLE MANIFOLD

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Abstract. We define Hitchin's moduli space $\mathcal{M}^{\mathrm{Hitchin}}(P)$ for a principal bundle P, whose structure group is a compact semisimple Lie group K, over a compact non-orientable Riemannian manifold M. We show that the Donaldson-Corlette correspondence, which identifies Hitchin's moduli space with the moduli space of flat $K^{\mathbb{C}}$-connections, remains valid when M is non-orientable. This enables us to study Hitchin's moduli space both by gauge theoretical methods and algebraically by using representation varieties. If the orientable double cover $\tilde{M}$ of M is a Kähler manifold with odd complex dimension and if the Kähler form is odd under the non-trivial deck transformation τ on $\tilde{M}$, Hitchin's moduli space $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})$ of the pull-back bundle $\tilde{P} \to \tilde{M}$ has a hyper-Kähler structure and admits an involution induced by τ. The fixed-point set $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^{\tau}$ is symplectic or Lagrangian with respect to various symplectic structures on $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})$. We show that there is a local diffeomorphism from $\mathcal{M}^{\mathrm{Hitchin}}(P)$ to $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^{\tau}$. We compare the gauge theoretical constructions with the algebraic approach using representation varieties.

1. Introduction

Let M be a compact orientable Riemannian manifold and let K be a connected compact Lie group. Given a principal K-bundle $P \to M$, let $\mathcal{A}(P)$ be the space of connections and let $\mathcal{G}(P)$ be the group of gauge transformations on P. Consider Hitchin's equations

$$F_A - \tfrac{1}{2}[\psi,\psi] = 0, \quad d_A\psi = 0, \quad d_A^*\psi = 0 \tag{1.1}$$

on the pairs $(A,\psi) \in \mathcal{A}(P) \times \Omega^1(M, \mathrm{ad}P)$. Hitchin's moduli space $\mathcal{M}^{\mathrm{Hitchin}}(P)$ is the set of space of solutions (A,ψ) to (1.1) modulo $\mathcal{G}(P)$ [Hi, S1]. On the other hand, let $G = K^{\mathbb{C}}$ be the complexification of K and let $P^{\mathbb{C}} = P \times_K G$, which is a principal bundle with structure group G. The moduli space $\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})$ of flat G-connections on $P^{\mathbb{C}}$, also known as the de Rham moduli space, is the space of flat reductive connections of $P^{\mathbb{C}}$ modulo $\mathcal{G}(P)^{\mathbb{C}} \cong \mathcal{G}(P^{\mathbb{C}})$. A theorem of Donaldson [D2] and Corlette [C] states that the moduli spaces $\mathcal{M}^{\mathrm{Hitchin}}(P)$ and $\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})$ are homeomorphic. The smooth part of $\mathcal{M}^{\mathrm{Hitchin}}(P)$ is a Kähler manifold with a complex structure $\bar{J}$ induced by that on G.

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Suppose in addition that M is a Kähler manifold. Then there is another complex structure $\bar{I}$ on $\mathcal{M}^{\mathrm{Hitchin}}(P)$ induced by that on M, and a third one given by $\bar{K}=\bar{I}\bar{J}$. The three complex structures $\bar{I},\bar{J},\bar{K}$ and their corresponding Kähler forms $\bar{\omega}_I,\bar{\omega}_J,\bar{\omega}_K$ form a hyper-Kähler structure on (the smooth part of) $\mathcal{M}^{\mathrm{Hitchin}}(P)$ [Hi, S1]. This hyper-Kähler structure comes from an infinite dimensional version of hyper-Kähler quotient [HKLR] of the tangent bundle $T\mathcal{A}(P)$, which is hyper-Kähler, by the action of $\mathcal{G}(P)$, which is Hamiltonian with respect to each of the Kähler forms $\omega_I,\omega_J,\omega_K$ on $T\mathcal{A}(P)$. When M is a compact orientable surface, Hitchin's moduli space $\mathcal{M}^{\mathrm{Hitchin}}(P)$ is equal to the hyper-Kähler quotient $\mathcal{M}^{\mathrm{HK}}(P):=T\mathcal{A}(P)/\!\!/\!\!/_0\,\mathcal{G}(P)$ [Hi]. It plays an important role in mirror symmetry and geometric Langlands program [HT, KW]. When M is higher dimensional, $\mathcal{M}^{\mathrm{Hitchin}}(P)$ is a hyper-Kähler subspace in $\mathcal{M}^{\mathrm{HK}}(P)$ [S1].

For a compact Lie group K, the moduli space of flat K-connections on a compact orientable surface was already studied in a celebrated work of Atiyah and Bott [AB]. When M is a compact, nonorientable surface, the moduli space of flat K-connections was studied in [Ho, HL2] through an involution on the space of connections over its orientable double cover $\tilde{M}$, induced by lifting the deck transformation on $\tilde{M}$ to the pull-back $\tilde{P}\to\tilde{M}$ of the given K-bundle $P\to M$ so that the quotient of $\tilde{P}$ by the involution is the original bundle P itself. This involution acts trivially on the structure group K. If instead one considers an involution on the bundle over $\tilde{M}$ that acts nontrivially on the fibers (such as the complex conjugation), then the fixed points give rise to the moduli space of real or quaternionic vector bundles over a real algebraic curve. This was studied thoroughly in [BHH, Sch], for example when $K=U(n)$.

In this paper, we study Hitchin's equations on a non-orientable manifold. Let M be a compact connected non-orientable Riemannian manifold and let $P\to M$ be a principal K-bundle over M, where K is a compact connected Lie group. The de Rham moduli space $\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})$, i.e., the moduli space of flat connections on $P^{\mathbb{C}}$, does not depend on the orientability of M. On the other hand, Hitchin's equations (1.1) on the pairs $(A,\psi)\in\mathcal{A}(P)\times\Omega^1(M,\mathrm{ad}P)$ still make sense (see subsection 2.2). We define Hitchin's moduli space $\mathcal{M}^{\mathrm{Hitchin}}(P)$ as the quotient of the space of pairs (A,ψ) satisfying (1.1) by the group $\mathcal{G}(P)$ of gauge transformations on P. Using the oriented cover $\pi\colon\tilde{M}\to M$ and the involutions induced by the non-trivial deck transformation τ on $\tilde{M}$, we show (in Theorem 2.2) that the homeomorphism $\mathcal{M}^{\mathrm{Hitchin}}(P)\cong\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})$ remains valid when M is non-orientable.

If the oriented cover $\tilde{M}$ of M is a Kähler manifold, then for the pull-back bundle $\tilde{P}:=\pi^*P$ over $\tilde{M}$, Hitchin's moduli space $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})$ is hyper-Kähler with complex structures $\bar{I},\bar{J},\bar{K}$ and Kähler forms $\bar{\omega}_I,\bar{\omega}_J,\bar{\omega}_K$. If the Kähler form ω on $\tilde{M}$ satisfies $\tau^*\omega=-\omega$ (the complex dimension of $\tilde{M}$ is odd so that τ is orientation reversing), then τ induces an involution (still denoted by τ) on $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})$ that satisfies $\tau^*\bar{\omega}_I = -\bar{\omega}_I$, $\tau^*\bar{\omega}_J = \bar{\omega}_J$ and $\tau^*\bar{\omega}_K = -\bar{\omega}_K$. Consequently, the fixed-point set $(\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P}))^\tau$ is Lagrangian in $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})$ with respect to $\bar{\omega}_I, \bar{\omega}_K$ and symplectic with respect to $\bar{\omega}_J$. This is known as an (A,B,A)-brane in [KW]. We discover that Hitchin's moduli space $\mathcal{M}^{\mathrm{Hitchin}}(P)$ (where M is non-orientable) is related to $(\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P}))^\tau$ by a local diffeomorphism. Our main results are summarized in the following main theorem. For simplicity, we restrict to certain smooth parts $\mathcal{M}^{\mathrm{Hitchin}}(P)^\circ$, $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^\circ$ and $\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})^\circ$ of the respective spaces (see subsection 2.3 for details).

Theorem 1.1. *Let M be a compact non-orientable manifold and let $\pi\colon \tilde{M} \to M$ be its oriented cover on which there is a non-trivial deck transformation τ. Let K be a compact connected Lie group. Given a principal K-bundle $P \to M$, let $\tilde{P} = \pi^*P$ be its pull-back to $\tilde{M}$. Suppose that $\tilde{M}$ is a Kähler manifold of odd complex dimension and the Kähler form ω on $\tilde{M}$ satisfies $\tau^*\omega = -\omega$. Then*
(1) $\mathcal{M}^{\mathrm{Hitchin}}(P)^\circ = \mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})^\circ /\!\!/_0 \mathcal{G}(P)$, which is a symplectic quotient.
(2) $(\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^\circ)^\tau$ is Kähler and totally geodesic in $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^\circ$ with respect to $\bar{J}, \bar{\omega}_J$ and totally real and Lagrangian with respect to $\bar{I}, \bar{K}$ and $\bar{\omega}_I, \bar{\omega}_K$.
(3) there is a local Kähler diffeomorphism from $\mathcal{M}^{\mathrm{Hitchin}}(P)^\circ$ to $(\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^\circ)^\tau$.

The theorem of Donaldson and Corlette and its counterpart in the non-orientable setup (Theorem 2.2) enable us to identify Hitchin's moduli space associated to an orientable or non-orientable manifold with the moduli space of flat connections and therefore the representation varieties. Let $\mathit{\Gamma}$ be a finitely generated group and let G be a connected complex semi-simple Lie group. The representation variety, $\mathrm{Hom}(\mathit{\Gamma}, G)/\!\!/G := \mathrm{Hom}^{\mathrm{red}}(\mathit{\Gamma}, G)/G$, is the quotient of the space of reductive homomorphisms from $\mathit{\Gamma}$ to G by the conjugation action of G. When $\mathit{\Gamma}$ is the fundamental group of a compact manifold M, the representation variety is also called the Betti moduli space of M; it is homeomorphic to the union of the de Rham moduli spaces $\mathcal{M}^{\mathrm{dR}}(P)$ associated to principal K-bundles $P \to M$ of various topology. When M is non-orientable, let $\tilde{\mathit{\Gamma}}$ be the fundamental group of the oriented cover $\tilde{M}$. Then there is a short exact sequence $1 \to \tilde{\mathit{\Gamma}} \to \mathit{\Gamma} \to \mathbb{Z}_2 \to 1$ and τ acts an involution on the representation variety $\mathrm{Hom}(\tilde{\mathit{\Gamma}}, G)/\!\!/G$ (Lemma 3.3). We study the relation of representation varieties associated to $\mathit{\Gamma}$ and $\tilde{\mathit{\Gamma}}$ from an algebraic point of view. Let $PG = G/Z(G)$, where $Z(G)$ is the center of G. Our main results are summarized in the following theorem.

Theorem 1.2. *Let G be a connected complex semi-simple Lie group. Let M be a compact non-orientable manifold and let $\tilde{M}$ be its oriented cover on which there is a non-trivial deck transformation τ. Denote $\mathit{\Gamma} = \pi_1(M)$ and $\tilde{\mathit{\Gamma}} = \pi_1(\tilde{M})$ with some chosen base points. Then*
(1) there exists a continuous map L from $(\mathrm{Hom}^{\mathrm{good}}(\tilde{\mathit{\Gamma}}, G)/G)^\tau$ to $Z(G)/2Z(G)$.

Consequently, $(\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma},G)/G)^\tau = \bigcup_{r\in Z(G)/2Z(G)}\mathcal{N}_r^{\mathrm{good}}$*, where* $\mathcal{N}_r^{\mathrm{good}}$ *is the preimage of* $r\in Z(G)/2Z(G)$*.*
(2) there exists a $|Z(G)/2Z(G)|$*-sheeted Galois covering map from* $\mathrm{Hom}_\tau^{\mathrm{good}}(\Gamma,G)/G$ *to* $\mathcal{N}_0^{\mathrm{good}}$*.*
In particular, if $|Z(G)|$ *is odd, then there exists a bijection from* $\mathrm{Hom}_\tau^{\mathrm{good}}(\Gamma,G)/G$ *to* $(\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma},G)/G)^\tau$*. The above statements are true if* $\mathrm{Hom}^{\mathrm{good}}$ *is replaced by* $\mathrm{Hom}^{\mathrm{irr}}$*.*
If in addition $M=\Sigma$ *is a compact non-orientable surface and* G *is simple and simply connected, then*
(3) there exists a surjective map from $(\mathrm{Hom}^{\mathrm{irr}}(\tilde{\Gamma},G)/G)^\tau$ *to* $\mathrm{Hom}_\tau^{\mathrm{irr}}(\Gamma,PG)/PG$ *that maps* $\mathcal{N}_r^{\mathrm{irr}}$ *to flat* PG*-bundles on* Σ *whose topological type is* $r\in Z(G)/2Z(G)\cong H^2(\Sigma,Z(G))$*. In particular,* $\mathcal{N}_0^{\mathrm{irr}}$ *maps to the topologically trivial flat* PG*-bundles on* Σ*.*

Here $\mathrm{Hom}^{\mathrm{good}}$ denotes the "good" part of the space of homomorphisms that are reductive and whose stabilizer is $Z(G)$, whereas $\mathrm{Hom}^{\mathrm{irr}}$ is the space of homomorphisms whose composition with the adjoint representation of G is an irreducible representation (see subsection 3.1 for details). $\mathrm{Hom}_\tau^{\mathrm{good}}(\Gamma,G)/\!\!/G$ is not smooth in general, but contains a smooth part $(\mathcal{M}^{\mathrm{flat}}(P^{\mathbb{C}}))^\circ$ (upon identification of moduli spaces). By parts (1) and (2) of the theorem, there is a local homeomorphism $\mathrm{Hom}_\tau^{\mathrm{good}}(\Gamma,G)/G\to(\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma},G)/G)^\tau$ (see also Corollary 3.7), which in fact restricts to the local diffeomorphism $\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})^\circ\to(\mathcal{M}^{\mathrm{dR}}(\tilde{P}^{\mathbb{C}})^\circ)^\tau$ in part (3) of Theorem 1.1 but is now more accurately described using representation varieties. Also, for $\phi\in\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma},G)$ such that $[\phi]\in\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma},G)/G$ is fixed by τ, $L([\phi])$ is the obstruction of extending ϕ to a representation of Γ. In the gauge-theoretic language, ϕ corresponds to a flat connection on $\tilde{M}$ and represents a point fixed by τ in the de Rham moduli space $\mathcal{M}^{\mathrm{dR}}(\tilde{P}^{\mathbb{C}})$, while extension of ϕ to Γ means that the flat connection on $\tilde{M}$ is the pull-back of one on M. Flat connections on $\tilde{M}$ that are not pull-backs from M correspond to flat PG-bundles over M (where $PG=G/Z(G)$). This is shown in part (3) of Theorem 1.2 and then discussed in greater generality in the last section.

The rest of this paper is organized as follows. In Section 2, we review the basic setup in the orientable case and establish the counterpart of the Donaldson-Corlette theorem for bundles over non-orientable manifolds. We then study finite dimensional symplectic and hyper-Kähler manifolds with an involution and apply the results to the gauge theoretical setting to prove Theorem 1.1. In Section 3, we study flat G-connections by representation varieties. We show that a flat connection on M is reductive if and only if its pull-back to $\tilde{M}$ is reductive. We then define the continuous map in part (1) of Theorem 1.2 and prove the rest of the theorem. In Section 4, we relate the components $\mathcal{N}_r^{\mathrm{good}}$ $(r\neq 0)$ in Theorem 1.2 to G-bundles over $\tilde{M}$ admitting an involution up to $Z(G)$.

During the revision of this paper, we came across a few related works. We would like to thank O. Gacía-Prada for pointing out to us the paper [BGH], where they also work with the space of connections over a complex manifold with an anti-holomorphic involution. Their involution flips the structure group G (giving rise to a real form), which induces a different involution on the space of connections, and thus resulting in a different fixed-point set. To study the moduli space of G-bundles over a non-orientable manifold, our involution on the structure group G is the identity map. In a more recent paper [BS1], which partly overlaps with part (2) of our Theorem 1.1 when $\tilde{M}$ is a surface, Baraglia and Schaposnik observed that any antiholomorphic involution on an Riemann surface induces an involution on Hitchin's moduli space, and that the fixed-point set of the involution is Lagrangian or complex with respect to different symplectic or complex structures of the hyper-Kähler structure. They also explored further properties in relation to Hitchin's fibration and spectral data. While we were finalising our paper, the paper [BS2] appeared, which studies the different types of real structures on the moduli space in more detail.

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2. The gauge-theoretic perspective

2.1. **Basic setup in the orientable case.** Let K be a connected compact Lie group and let $G = K^{\mathbb{C}}$ be its complexification. Given a principal K-bundle P over a compact orientable manifold M, $P^{\mathbb{C}} = P \times_K G$ is a principal bundle whose structure group is G. The set $\mathcal{A}(P)$ of connections on P is an affine space modeled on $\Omega^1(M, \mathrm{ad}P)$. At each $A \in \mathcal{A}(P)$, the tangent space is $T_A\mathcal{A}(P) \cong \Omega^1(M, \mathrm{ad}P)$. The total space of the tangent bundle over $\mathcal{A}(P)$ is $T\mathcal{A}(P) = \mathcal{A}(P) \times \Omega^1(M, \mathrm{ad}P)$. At $(A, \psi) \in T\mathcal{A}(P)$, the tangent space is $T_{(A,\psi)}T\mathcal{A}(P) \cong \Omega^1(M, \mathrm{ad}P)^{\oplus 2}$. There is a translation invariant complex structure J on $T\mathcal{A}(P)$ given by $J(\alpha, \varphi) = (\varphi, -\alpha)$. The space $T\mathcal{A}(P)$ can be naturally identified with $\mathcal{A}(P^{\mathbb{C}})$, the set of connections on $P^{\mathbb{C}} \to M$, via $(A, \psi) \mapsto A - \sqrt{-1}\psi$, under which J corresponds to the complex structure on $\mathcal{A}(P^{\mathbb{C}})$ induced by $G = K^{\mathbb{C}}$. The covariant derivative on $\Omega^\bullet(M, \mathrm{ad}P^{\mathbb{C}})$ is $D := d_A - \sqrt{-1}\psi$, where d_A denotes the covariant derivative of $A \in \mathcal{A}(P)$ and ψ acts by bracket.

The group of gauge transformations on P is $\mathcal{G}(P) \cong \Gamma(M, \mathrm{Ad}P)$. It acts on $\mathcal{A}(P)$ via $A \mapsto g \cdot A$, where $d_{g\cdot A} = g \circ d_A \circ g^{-1}$ and on $T\mathcal{A}(P)$ via $g\colon (A, \psi) \mapsto (g\cdot A, \mathrm{Ad}_g \psi)$. Since the action of $\mathcal{G}(P)$ on $T\mathcal{A}(P)$ preserves J, there is a holomorphic $\mathcal{G}(P)^{\mathbb{C}}$ action on $(T\mathcal{A}(P), J)$. In fact, the complexification $\mathcal{G}(P)^{\mathbb{C}}$ can be naturally identified with $\mathcal{G}(P^{\mathbb{C}}) \cong \Gamma(M, \mathrm{Ad}P^{\mathbb{C}})$, and the action of $\mathcal{G}(P^{\mathbb{C}})$ on $T\mathcal{A}(P)$ corresponds to the complex gauge transformations on $\mathcal{A}(P^{\mathbb{C}})$, i.e., $g\in\mathcal{G}(P^{\mathbb{C}})\colon D\mapsto g\circ D\circ g^{-1}$. Let

$$\begin{aligned}\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}}) &= \{A-\sqrt{-1}\psi\in\mathcal{A}(P^{\mathbb{C}}) : F_{A-\sqrt{-1}\psi}=0\}\\ &= \big\{(A,\psi)\in T\mathcal{A} : F_A-\tfrac{1}{2}[\psi,\psi]=0, d_A\psi=0\big\}\end{aligned}$$

be the set of flat connections on $P^{\mathbb{C}}$. Since the vanishing of $F_{A-\sqrt{-1}\psi}$ is a holomorphic condition, $\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})$ is a complex subset of $\mathcal{A}(P^{\mathbb{C}})$; it is also invariant under $\mathcal{G}(P^{\mathbb{C}})$. The holonomy group $\mathrm{Hol}(A)$ of $A\in\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})$ can be identified as a subgroup of G, up to a conjugation in G. A flat connection A on $P^{\mathbb{C}}$ is *reductive* if the closure of $\mathrm{Hol}(A)$ in G is contained in the Levi subgroup of any parabolic subgroup containing $\mathrm{Hol}(A)$; let $\mathcal{A}^{\mathrm{flat,red}}(P^{\mathbb{C}})$ be the set of such. It can be shown that a flat connection is reductive if and only if its orbit under $\mathcal{G}(P^{\mathbb{C}})$ is closed [C]. The *de Rham moduli space*, or the moduli space of reductive flat connections on $P^{\mathbb{C}}$, is

$$\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})=\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})/\!\!/\mathcal{G}(P^{\mathbb{C}})=\mathcal{A}^{\mathrm{flat,red}}(P^{\mathbb{C}})/\mathcal{G}(P^{\mathbb{C}}).$$

It has an induced complex structure denoted by $\bar{J}$.

Assume that M has a Riemannian structure and choose an invariant inner product $(\cdot,\cdot)$ on the Lie algebra $\mathfrak{k}$ of K. Then there is a symplectic structure on $T\mathcal{A}(P)$, with which J is compatible, given by

$$\omega_J((\alpha_1,\varphi_1),(\alpha_2,\varphi_2))=\int_M(\varphi_2,\wedge *\alpha_1)-(\varphi_1,\wedge *\alpha_2),$$

where $\alpha_1,\alpha_2,\varphi_1,\varphi_2\in\Omega^{1,0}(M,\mathrm{ad}P)$, such that $(T\mathcal{A}(P),\omega_J)$ is Kähler. The subset $\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})$ is Kähler in $\mathcal{A}(P^{\mathbb{C}})\cong T\mathcal{A}(P)$. We identify the Lie algebra $\mathrm{Lie}(\mathcal{G}(P))\cong\Omega^0(M,\mathrm{ad}P)$ with its dual by the inner product on $\Omega^0(M,\mathrm{ad}P)$. The action of $\mathcal{G}(P)$ on $(T\mathcal{A}(P),\omega_J)$ is Hamiltonian, with moment map

$$\mu_J(A,\psi)=d_A^*\psi\in\Omega^0(M,\mathrm{ad}P).$$

Let

$$\begin{aligned}&\mathcal{A}^{\mathrm{Hitchin}}(P)=\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})\cap\mu_J^{-1}(0)\\ &=\big\{(A,\psi)\in T\mathcal{A} : F_A-\tfrac{1}{2}[\psi,\psi]=0, d_A\psi=0, d_A^*\psi=0\big\},\end{aligned}$$

the set of pairs (A,ψ) satisfying Hitchin's equations (1.1), and let $\mathcal{M}^{\mathrm{Hitchin}}(P)=\mathcal{A}^{\mathrm{Hitchin}}(P)/\mathcal{G}(P)$ be *Hitchin's moduli space*. A theorem of Donaldson [D2] and Corlette [C] states that if M is compact and if the structure group G is semisimple, then $\mathcal{M}^{\mathrm{Hitchin}}(P)\cong\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})$.

Suppose that M is a compact Kähler manifold of complex dimension n and let ω be the Kähler form on M. Then there is a complex structure on $T\mathcal{A}(P)$ given by

$$I\colon(\alpha,\varphi)\mapsto\frac{1}{(n-1)!}*(\omega^{n-1}\wedge(\alpha,-\varphi))=\frac{1}{(n-1)!}\Lambda^{n-1}(*\alpha,-*\varphi),$$

where $(\alpha,\varphi)\in\Omega^1(M,\mathrm{ad}P)^{\oplus 2}\cong T_{(A,\psi)}T\mathcal{A}(P)$ and $\Lambda\colon\Omega^\bullet(M,\mathrm{ad}P)\to\Omega^{\bullet-2}(M,\mathrm{ad}P)$ is the contraction by ω. With respect to I, we have $T^{1,0}_{(A,\psi)}T\mathcal{A}(P)\cong\Omega^{0,1}(M,\mathrm{ad}P^{\mathbb{C}})\oplus$

$\Omega^{1,0}(M,\mathrm{ad}P^{\mathbb{C}})$ for any $(A,\psi)\in T\mathcal{A}(P)$. This complex structure I is compatible with a symplectic form ω_I on $T\mathcal{A}(P)$ given by

$$\omega_I((\alpha_1,\varphi_1),(\alpha_2,\varphi_2))=\int_M\frac{\omega^{n-1}}{(n-1)!}\wedge\big((\alpha_1,\wedge\alpha_2)-(\varphi_1,\wedge\varphi_2)\big),$$

where $\alpha_1,\alpha_2,\varphi_1,\varphi_2\in\Omega^1(M,\mathrm{ad}P)$. The action of $\mathcal{G}(P)$ on $T\mathcal{A}(P)$ is also Hamiltonian with respect to ω_I and the moment map is

$$\mu_I(A,\psi)=\Lambda\,(F_A-\tfrac{1}{2}[\psi,\psi])\in\Omega^0(M,\mathrm{ad}P),$$

where $F_A\in\Omega^2(M,\mathrm{ad}P)$ is the curvature of A. Since the action of $\mathcal{G}(P)$ on $T\mathcal{A}(P)$ preserves I, there is a holomorphic $\mathcal{G}(P^{\mathbb{C}})$ action on $(T\mathcal{A}(P),I)$. For any $(A,\psi)\in T\mathcal{A}(P)$, write $\psi=\sqrt{-1}(\phi-\phi^*)$, where $\phi\in\Omega^{1,0}(M,\mathrm{ad}P^{\mathbb{C}})$, $\phi^*\in\Omega^{0,1}(M,\mathrm{ad}P^{\mathbb{C}})$. Here $\phi\mapsto\phi^*$ is induced by the conjugation on $G=K^{\mathbb{C}}$ preserving the compact form K. Then $D=d_A-\sqrt{-1}\psi=D'+D''$, where $D'=\partial_A-\phi^*$, $D''=\bar\partial_A+\phi$. The action of $\mathcal{G}(P^{\mathbb{C}})$ on $T\mathcal{A}(P)\cong\mathcal{A}(P^{\mathbb{C}})$ can be described by $g\in\mathcal{G}(P^{\mathbb{C}})\colon D''\mapsto g\circ D''\circ g^{-1}$.

Let $\mathcal{A}^{\mathrm{Higgs}}(P^{\mathbb{C}})$ be the set of Higgs pairs (A,ϕ), i.e., $A\in\mathcal{A}(P)$ and $\phi\in\Omega^{1,0}(M,\mathrm{ad}P^{\mathbb{C}})$ satisfying $(D'')^2=0$, or

$$\bar\partial_A^2=0,\quad \bar\partial_A\phi=0,\quad [\phi,\phi]=0.$$

Then $\mathcal{A}^{\mathrm{Higgs}}(P^{\mathbb{C}})$ is a Kähler subspace of $\mathcal{A}(P^{\mathbb{C}})\cong T\mathcal{A}(P)$ respect to I. Let $\mathcal{A}^{\mathrm{sst}}(P^{\mathbb{C}})$ be the set of semi-stable Higgs pairs and let $\mathcal{A}^{\mathrm{pst}}(P^{\mathbb{C}})$ be the set polystable Higgs pairs. (The notions of stable, semistable and polystable Higgs pairs were introduced in [Hi, S2, S3].) The *moduli space of polystable Higgs pairs* or the *Dolbeault moduli space* is

$$\mathcal{M}^{\mathrm{Dol}}(P^{\mathbb{C}})=(\mathcal{A}^{\mathrm{Higgs}}(P^{\mathbb{C}})\cap\mathcal{A}^{\mathrm{sst}}(P^{\mathbb{C}}))/\!\!/\mathcal{G}(P^{\mathbb{C}})=(\mathcal{A}^{\mathrm{Higgs}}(P^{\mathbb{C}})\cap\mathcal{A}^{\mathrm{pst}}(P^{\mathbb{C}}))/\mathcal{G}(P^{\mathbb{C}}).$$

It has a complex structure induced by I. It can be shown [S2, Lemma 1.1] that $\mathcal{A}^{\mathrm{Hitchin}}(P)=\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})\cap\mu_J^{-1}(0)=\mathcal{A}^{\mathrm{Higgs}}(P^{\mathbb{C}})\cap\mu_I^{-1}(0)$. A theorem of Hitchin [Hi] and Simpson [S1] states that if M is compact and Kähler and the bundle P has vanishing first and second Chern classes, then $\mathcal{M}^{\mathrm{Hitchin}}(P)\cong\mathcal{M}^{\mathrm{Dol}}(P^{\mathbb{C}})$.

There is a third complex structure on $T\mathcal{A}(P)$ defined by

$$K=IJ=-JI\colon(\alpha,\varphi)\mapsto\frac{1}{(n-1)!}*(\omega^{n-1}\wedge(\varphi,\alpha))=\frac{1}{(n-1)!}\Lambda^{n-1}(*\varphi,*\alpha),$$

which is compatible with the symplectic form

$$\omega_K((\alpha_1,\varphi_1),(\alpha_2,\varphi_2))=\int_M\frac{\omega^{n-1}}{(n-1)!}\wedge\big((\alpha_1,\wedge\varphi_2)-(\alpha_2,\wedge\varphi_1)\big).$$

The action of $\mathcal{G}(P)$ on $T\mathcal{A}(P)$ is Hamiltonian with respect to ω_K and the moment map is

$$\mu_K(A,\psi)=\Lambda\,(d_A\psi)\in\Omega^0(M,\mathrm{ad}P).$$

Moreover, the action preserves K and therefore extends to another holomorphic action of $\mathcal{G}(P)^{\mathbb{C}}$. The three complex structures I,J,K define a hyper-Kähler structure on $T\mathcal{A}(P)$. Since the action of $\mathcal{G}(P)$ on $T\mathcal{A}(P)$ is Hamiltonian with respect to all three symplectic forms, we have a hyper-Kähler moment map $\mu = (\mu_I, \mu_J, \mu_K)\colon T\mathcal{A}(P) \to (\Omega^0(M, \mathrm{ad}P))^{\oplus 3}$. The *hyper-Kähler quotient* [HKLR] is $\mathcal{M}^{\mathrm{HK}}(P) = \mu^{-1}(0)/\mathcal{G}(P)$, with complex structures $\bar{I}, \bar{J}, \bar{K}$ and symplectic forms $\bar{\omega}_I, \bar{\omega}_J, \bar{\omega}_K$. By the theorems of Donaldson-Corlette and of Hitchin-Simpson, the Hitchin moduli space $\mathcal{M}^{\mathrm{Hitchin}}(P)$ is a complex space with respect to both $\bar{I}$ and $\bar{J}$. Therefore $\mathcal{M}^{\mathrm{Hitchin}}(P)$ is a hyper-Kähler subspace in $\mathcal{M}^{\mathrm{HK}}(P)$ [Fu, Theorem 8.3.1].

When $M = \Sigma$ is an orientable surface, the map $\Lambda\colon \Omega^2(\Sigma, \mathrm{ad}P) \to \Omega^0(\Sigma, \mathrm{ad}P)$ is an isomorphism. So $\mathcal{A}^{\mathrm{Hitchin}}(P) = \mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}}) \cap \mu_J^{-1}(0) = \mathcal{A}^{\mathrm{Higgs}}(P^{\mathbb{C}}) \cap \mu_I^{-1}(0)$ coincides with $\mu^{-1}(0) = \mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0)$. Therefore the moduli spaces $\mathcal{M}^{\mathrm{Hitchin}}(P) \cong \mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}}) \cong \mathcal{M}^{\mathrm{Dol}}(P^{\mathbb{C}})$ coincide with the hyper-Kähler quotient $\mathcal{M}^{\mathrm{HK}}(P)$ [Hi].

2.2. **The Donaldson-Corlette correspondence in the non-orientable case.** Now suppose M is a compact non-orientable manifold. Let $\pi\colon \tilde{M} \to M$ be its oriented cover and let $\tau\colon \tilde{M} \to \tilde{M}$ be the non-trivial deck transformation. Given a principal K-bundle $P \to M$, let $\tilde{P} = \pi^* P \to \tilde{M}$ be its pull-back to $\tilde{M}$. Since $\pi \circ \tau = \pi$, the τ action can be lifted to $\tilde{P} = \tilde{M} \times_M P$ as a K-bundle involution (i.e., the lifted involution commutes with the right K-action on $\tilde{P}$), and hence to the associated bundles $\mathrm{Ad}\tilde{P}$ and $\mathrm{ad}\tilde{P}$. Consequently, τ acts on the space of connections $\mathcal{A}(\tilde{P})$ by pull-back $A \mapsto \tau^* A$ and on the group of gauge transformations $\mathcal{G}(\tilde{P})$ by $g \mapsto \tau^* g := \tau^{-1} \circ g \circ \tau$. The τ-invariant subsets are $(\mathcal{A}(\tilde{P}))^\tau \cong \mathcal{A}(P)$ and $(\mathcal{G}(\tilde{P}))^\tau \cong \mathcal{G}(P)$. In fact, the inclusion map $\mathcal{A}(P) \hookrightarrow \mathcal{A}(\tilde{P})$ onto the τ-invariant part is the pull-back map of connections on P to those on $\tilde{P}$. Since $\mathcal{A}(\tilde{P})$ is an affine space modeled on $\Omega^1(\tilde{M}, \mathrm{ad}\tilde{P})$, the differential τ_* of $\tau\colon \mathcal{A}(\tilde{P})(P) \to \mathcal{A}(\tilde{P})$ can be identified with a linear involution on $\Omega^1(\tilde{M}, \mathrm{ad}\tilde{P})$ given by $\alpha \mapsto \tau^* \alpha$.

Consider the tangent bundle $T\mathcal{A}(\tilde{P}) = \mathcal{A}(\tilde{P}) \times \Omega^1(\tilde{M}, \mathrm{ad}\tilde{P})$ of $\mathcal{A}(\tilde{P})$. It has a τ-action given by $\tau\colon (A, \psi) \mapsto (\tau^* A, \tau^* \psi)$, which is holomorphic with respect to the complex structure J. Therefore the fixed point set $(T\mathcal{A}(\tilde{P}))^\tau \cong T\mathcal{A}(P)$ is a complex subspace in $T\mathcal{A}(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$. A Riemannian structure on M induces a τ-invariant one on $\tilde{M}$ and hence on $T\mathcal{A}(\tilde{P})$, such that $\tau\colon T\mathcal{A}(\tilde{P}) \to T\mathcal{A}(\tilde{P})$ is an isometry. Since τ also acts holomorphically on $\mathcal{A}(\tilde{P}^{\mathbb{C}}) \cong T\mathcal{A}(\tilde{P})$, $(T\mathcal{A}(\tilde{P}))^\tau$ is a Kähler and totally geodesic subspace in $T\mathcal{A}(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$. Moreover, $\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}}) \cong (\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}}))^\tau$ is also Kähler and totally geodesic in $\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})$. On the other hand, we still have the subspace

$$\mathcal{A}^{\mathrm{Hitchin}}(P) = \{(A, \psi) \in T\mathcal{A} : F_A - \tfrac{1}{2}[\psi, \psi] = 0, d_A \psi = 0, d_A^* \psi = 0\},$$

where d_A^* can be defined as the (formal) adjoint of d_A with respect to the inner products on $\Omega^\bullet(M, \mathrm{ad}P)$. Alternatively, d_A^* is the first order differential operator on M such that on any orientable open set in M, $d_A^* = *^{-1} d_A *$; the latter is actually independent of the choice of local orientation. Yet another but related way to explain the operator d_A^* is to consider the Hodge star operator $*$ on a non-orientable manifold M as a map from differential forms to those valued in the orientable bundle over M. Since the latter is a flat real line bundle, $d_A^* = *^{-1} d_A *$ maps $\Omega^1(M, \mathrm{ad}P)$ to $\Omega^0(M, \mathrm{ad}P)$. Finally, d_A^* can be defined as $(\pi^*)^{-1} \circ d_{\pi^* A}^* \circ \pi^*$. Here $d_{\pi^* A}^* = *^{-1} d_{\pi^* A} *$ holds globally on $\tilde{M}$ and $\pi^* \colon \Omega^\bullet(M, \mathrm{ad}P) \to \Omega^\bullet(\tilde{M}, \mathrm{ad}\tilde{P})$ is injective. It is clear that $\mathcal{A}^{\mathrm{Hitchin}}(P) = (\mathcal{A}^{\mathrm{Hitchin}}(\tilde{P}))^\tau$.

We summarize the above discussion in the following lemma.

Lemma 2.1. *Given a compact non-orientable manifold M with its the oriented cover $\pi\colon \tilde{M} \to M$ and a principal K-bundle $P \to M$, the non-trivial deck transformation τ on $\tilde{M}$ lifts to an involution (also denoted by τ) on $\tilde{P} = \pi^* P$ and acts as involutions on the space of connections $\mathcal{A}(\tilde{P})$ and on $T\mathcal{A}(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$. Moreover, the τ-invariant subspaces $\mathcal{A}(\tilde{P}^{\mathbb{C}})^\tau \cong \mathcal{A}(P^{\mathbb{C}})$ and $\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^\tau \cong \mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})$ are Kähler and totally geodesic subspaces in $\mathcal{A}(\tilde{P}^{\mathbb{C}}) \cong T\mathcal{A}(\tilde{P})$ and $\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})$, respectively.*

The notion of reductive connections on P does not depend on the orientability of M, and we still have the moduli space of flat connections $\mathcal{M}^{\mathrm{flat}}(P^{\mathbb{C}}) = \mathcal{A}^{\mathrm{flat,red}}(P^{\mathbb{C}})/\mathcal{G}(P^{\mathbb{C}})$. Let $\mathcal{M}^{\mathrm{Hitchin}}(P) = \mathcal{A}^{\mathrm{Hitchin}}(P)/\mathcal{G}(P)$ be Hitchin's moduli space. Our main goal is to prove a correspondence between the moduli spaces $\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})$ and $\mathcal{M}^{\mathrm{Hitchin}}(P)$ when M is non-orientable.

Theorem 2.2. *Let M be a compact non-orientable Riemannian manifold. Then for every reductive flat connection D on $P^{\mathbb{C}}$, there exists a gauge transformation $g \in \mathcal{G}(P^{\mathbb{C}})$ (unique up to $\mathcal{G}(P)$ and the stabilizer of D) such that $g \cdot D = d_A - \sqrt{-1}\psi$ with $(A, \psi) \in \mathcal{A}^{\mathrm{Hitchin}}(P)$. As a consequence, we have a homeomorphism $\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}}) \cong \mathcal{M}^{\mathrm{Hitchin}}(P)$.*

Equivalently, there exists a unique reduction of structure group from G to K admitting a solution to Hitchin's equations. This is the non-orientable analog of the Donaldson-Corlette theorem in [D2, C]. The strategy of the proof is to use Corlette's results on the orientable double cover and show that the subset of τ-invariant reductive flat connections has the property that the reduction of structure group guaranteed by Corlette's theorem is also τ-invariant.

Recall Corlette's flow equations on the space of flat connections. Let $D = d_A - \sqrt{-1}\psi$ be a flat connection of the $G = K^{\mathbb{C}}$ bundle $\tilde{P}^{\mathbb{C}} \to \tilde{M}$. Then the flow equations are

$$\frac{\partial D}{\partial t} = -D \upmu_J(D), \tag{2.1}$$

where $\mu_J(D) = \upmu_J(A, \psi) = d_A^* \psi$. Equivalently, one can look for a flow of the form $g(t) \cdot D_0$ and solve for $g(t) \in \mathcal{G}(\tilde{P}^{\mathbb{C}})$ using (cf. [C, p. 369])

$$\frac{\partial g}{\partial t} g^{-1} = -\sqrt{-1} \upmu_J(g \cdot D_0). \tag{2.2}$$

It is easy to check that the right-hand sides of (2.1) and (2.2) define τ-invariant vector fields on $\mathcal{A}(\tilde{P}^{\mathbb{C}})$ and $\mathcal{G}(\tilde{P}^{\mathbb{C}})$, respectively. Corlette shows in [C] that we have existence and uniqueness of solutions to (2.1) and (2.2) for all time. If the initial condition is a reductive flat connection, then there is a sequence converging to a solution to $\mu_J(D) = 0$. Also, the limit is gauge equivalent to the initial flat reductive connection [C].

The fact that these flows preserve the subset of τ-invariant connections and τ-invariant gauge transformations follows from the next lemma.

Lemma 2.3. *Let X be a manifold and Y a (possibly singular) subset. Let $\tau\colon X \to X$ be an involution with fixed point set X^τ. Let V be a τ-invariant vector field on X such that for each $x \in Y$ there exists a unique integral curve $x(t)$ for V for all $t \geq 0$ on the manifold X such that (a) $x(t) \in Y$ for all $t \geq 0$ and (b) $x(0) = x$. If $x \in Y^\tau$ then $x(t) \in Y^\tau$ for all $t \geq 0$.*

Proof. If V is τ-invariant then $\tau(x(t))$ is also an integral curve for V with initial condition $\tau(x(0)) = x(0)$. Since integral curves for V are unique, then $\tau(x(t)) = x(t)$ for all $t \geq 0$, i.e. $x(t) \in Y^\tau$. □

We now show that Theorem 2.2 then follows from Corlette's results applied to the orientable double cover. A flat connection on P is reductive if and only if the pull-back $\pi^* A$ is a flat reductive connection on $\tilde{P}$. (We defer the proof of this statement to Corollary 3.2.) Then we can either apply Lemma 2.3 to (2.1) with $X = \mathcal{A}(\tilde{P}^{\mathbb{C}})$ and $Y = \mathcal{A}^{\mathrm{flat,red}}(\tilde{P}^{\mathbb{C}})$ or to (2.2) with $X = Y = \mathcal{G}(\tilde{P}^{\mathbb{C}})$. Since the space $(\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}}))^\tau$ of τ-invariant connections is closed in $\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})$ and the space $(\mathcal{G}(\tilde{P}^{\mathbb{C}}))^\tau$ of τ-invariant gauge transformations is closed in $\mathcal{G}(\tilde{P}^{\mathbb{C}})$, Corlette's results on the limit of the flow restrict to the τ-invariant subset as well.

2.3. **The Hitchin moduli space and the hyper-Kähler quotient.** Now consider a compact non-orientable manifold M. Suppose its oriented cover $\tilde{M}$ is a Kähler manifold of complex dimension n. Let ω be the Kähler form on $\tilde{M}$. Throughout this subsection, we assume that n is odd and the deck transformation τ on $\tilde{M}$ is an anti-holomorphic involution such that $\tau^*\omega = -\omega$. Then $\tau^*\omega^n = -\omega^n$, which is consistent with the requirement that τ is orientation reversing. The τ-action on $T\mathcal{A}(\tilde{P}) = \mathcal{A}(\tilde{P}) \times \Omega^1(M, \mathrm{ad}P)$, $\tau\colon (A, \psi) \mapsto (\tau^*A, \tau^*\psi)$, is an isometry and its differential $\tau_*\colon \Omega^1(M, \mathrm{ad}P)^{\oplus 2} \to \Omega^1(M, \mathrm{ad}P)^{\oplus 2}$ is $\tau_*\colon (\alpha, \varphi) \mapsto (\tau^*\alpha, \tau^*\varphi)$. It is easy to see that $\tau_* \circ I = -I \circ \tau_*$ since τ reverses the orientation of M and that $\tau_* \circ K = -K \circ \tau_*$ since $K = IJ$. So τ acts as an anti-holomorphic involution with respect to both I and K, and $\tau^*\omega_I = -\omega_I$, $\tau^*\omega_K = -\omega_K$. Moreover, since the moment maps μ_I and μ_K on $T\mathcal{A}(\tilde{P})$ involve the contraction Λ by ω, they satisfy $\tau^*(\mu_I(A, \psi)) = -\mu_I(\tau^*A, \tau^*\psi)$, $\tau^*(\mu_K(A, \psi)) = -\mu_K(\tau^*A, \tau^*\psi)$ for all $(A, \psi) \in T\mathcal{A}(\tilde{P})$. The fixed point set $(\mathcal{A}(\tilde{P}))^\tau$ is totally real with respect to the complex structures I and K, and Lagrangian with respect to the symplectic forms ω_I and ω_K [Me, Du, OS].

We first study a finite dimensional model. Let (X,ω) be a finite dimensional symplectic manifold with a Hamiltonian action of a compact connected Lie group K and let $\mu\colon X\to\mathfrak{k}^*$ be the moment map. Suppose as in [OS], that there are involutions σ on X and τ on K such that $\sigma(k\cdot x)=\tau(k)\cdot\sigma(x)$ for all $k\in K$ and $x\in X$. Assume that X^σ is not empty. Then K^τ acts on X^σ. We note that τ acts on $\mathfrak{k}$, $\mathfrak{k}^*$, and K^τ is a closed Lie subgroup of K with Lie algebra $\mathfrak{k}^\tau$. Contrary to [OS], we assume that the action of (K,K^τ) on (X,X^σ) is symplectic, i.e, we have $\sigma^*\omega=\omega$ and $\sigma^*\mu=\tau\mu$. Then X^σ is a symplectic submanifold in X. Assume that 0 is a regular value of μ and that K acts on $\mu^{-1}(0)$ freely. Since σ preserves $\mu^{-1}(0)$, it descends to a symplectic involution $\bar\sigma$ on the (smooth) symplectic quotient $X/\!\!/_0K=\mu^{-1}(0)/K$ at level 0, and $(X/\!\!/_0K)^{\bar\sigma}$ is a symplectic submanifold.

Lemma 2.4. *In the above setting, the action of K^τ on X^σ is Hamiltonian and the symplectic quotient is $X^\sigma/\!\!/_0K^\tau=(\mu^{-1}(0)\cap X^\sigma)/K^\tau$. If $\mu^{-1}(0)\cap X^\sigma\neq\emptyset$, then there exists a symplectic local diffeomorphism from $X^\sigma/\!\!/_0K^\tau$ to $(X/\!\!/_0K)^{\bar\sigma}$.*

Proof. Let $\mathfrak{k}=\mathfrak{k}^\tau\oplus\mathfrak{q}$ such that $\tau=\pm1$ on $\mathfrak{k}^\tau$, $\mathfrak{q}$, respectively. It is clear that the action of K^τ on X^σ is Hamiltonian and the moment map μ_τ is the composition $X^\sigma\hookrightarrow X\to\mathfrak{k}^*\to(\mathfrak{k}^\tau)^*$. Since for any $x\in X^\sigma$, $\langle\mu(x),\mathfrak{q}\rangle=0$, we get $\mu_\tau^{-1}(0)=\mu^{-1}(0)\cap X^\sigma=(\mu^{-1}(0))^\sigma$. By the assumptions, 0 is a regular value of μ_τ, the action of K^τ on $\mu_\tau^{-1}(0)$ is free, and the symplectic quotient is $X^\sigma/\!\!/_0K^\tau=(\mu^{-1}(0)\cap X^\sigma)/K^\tau$.

For any $x\in X^\sigma$, the map $\mathfrak{k}\to T_xX$ intertwines τ on $\mathfrak{k}$ and σ on T_xX, and $T_x(K^\tau\cdot x)=(T_x(K\cdot x))^\sigma$. The inclusion $\mu_\tau^{-1}(0)\hookrightarrow\mu^{-1}(0)$ induces a natural map $X^\sigma/\!\!/_0K^\tau\to(X/\!\!/_0K)^{\bar\sigma}$, whose differentiation at $[x]$ is, after natural symplectic isomorphisms, the linear map $(T_x\mu^{-1}(0))^\sigma/(T_x(K\cdot x))^\sigma\to(T_x\mu^{-1}(0)/T_x(K\cdot x))^{\bar\sigma}$. The latter is clearly injective; to show surjectivity, we note that for any $V\in T_x\mu^{-1}(0)$, if $V+T_x(K\cdot x)\in(T_x\mu^{-1}(0)/T_x(K\cdot x))^{\bar\sigma}$, then it is the image of $\frac12(V+\sigma V)+(T_x(K\cdot x))^\sigma$. The map $X^\sigma/\!\!/_0K^\tau\to(X/\!\!/_0K)^{\bar\sigma}$ is a local diffeomorphism; it is symplectic because the above linear map is so for each $x\in\mu_\tau^{-1}(0)$. □

Now let X be a hyper-Kähler manifold with complex structures J_i and symplectic structures ω_i $(i=1,2,3)$. Suppose K acts on X and the action is Hamiltonian with respect to all ω_i. Let $\mu=(\mu_1,\mu_2,\mu_3)\colon X\to(\mathfrak{k}^*)^{\oplus3}$ be the hyper-Kähler moment map. Assume that there are involutions σ on X and τ on K such that $\sigma(k\cdot x)=\tau(k)\cdot\sigma(x)$ for all $k\in K$ and $x\in X$ and $\sigma^*J_i=(-1)^iJ_i$, $\sigma^*\omega_i=(-1)^i\omega_i$, $\sigma^*\mu_i=(-1)^i\tau\mu_i$ for $i=1,2,3$. So the action of (K,K^τ) on (X,X^σ) is symplectic with respect to ω_2 (as above) and anti-symplectic with respect to ω_1,ω_3 (as in [OS]). Then X^σ, if non-empty, is Kähler and totally geodesic in X with respect to

J_2, ω_2 and is totally real and Lagrangian with respect to J_1, ω_1 and J_3, ω_3. If 0 is a regular value of μ (i.e., 0 is a regular value of each μ_i) and that K acts on $\mu^{-1}(0)$ freely, then $X/\!\!/\!\!/_0 K = \mu^{-1}(0)/K$ is the (smooth) hyper-Kähler quotient at level 0, which has complex structures $\bar{J}_i$ and symplectic structures $\bar{\omega}_i$ $(i = 1, 2, 3)$ [HKLR].

Proposition 2.5. *In the above setting, let* $Y = \mu_1^{-1}(0) \cap \mu_3^{-1}(0)$. *Then*
1. Y *is a* σ*-invariant Kähler submanifold in* X *with respect to* J_2, ω_2 *and the symplectic quotient* $Y^\sigma /\!\!/_0 K^\tau = (\mu^{-1}(0))^\sigma / K^\tau$ *is Kähler;*
2. $(X/\!\!/\!\!/_0 K)^{\bar{\sigma}}$ *is Kähler and totally geodesic in* $X/\!\!/\!\!/_0 K$ *with respect to* $\bar{J}_2, \bar{\omega}_2$ *and is totally real and Lagrangian with respect to* $\bar{J}_1, \bar{J}_3$ *and* $\bar{\omega}_1, \bar{\omega}_3$*;*
3. if $(\mu^{-1}(0))^\sigma \neq \emptyset$*, there is a Kähler (with respect to* $\bar{J}_2, \bar{\omega}_2$*) local diffeomorphism* $Y^\sigma /\!\!/_0 K^\tau \to (X/\!\!/\!\!/_0 K)^{\bar{\sigma}}$.

Proof. 1&3. Let $\mu_c = \mu_3 + \sqrt{-1}\mu_1 \colon X \to \mathfrak{k}^{*\mathbb{C}}$. Then μ_c is holomorphic with respect to J_2 and is equivariant under the action of K. Since 0 is a regular value of μ_c, $Y = \mu_c^{-1}(0)$ is a smooth Kähler submanifold in X on which the action of K is Hamiltonian. Applying Lemma 2.4 to Y, we conclude that the action of K^τ on Y^σ is Hamiltonian and that $(\mu^{-1}(0))^\sigma / K^\tau = (\mu_2^{-1}(0) \cap Y^\sigma)/K^\tau = Y^\sigma /\!\!/_0 K^\tau$. Moreover, there is a local diffeomorphism from $Y^\sigma /\!\!/_0 K^\tau$ to $(Y/\!\!/_0 K)^{\bar{\sigma}} = (X/\!\!/\!\!/_0 K)^{\bar{\sigma}}$ which is symplectic. Since K^τ acts holomorphically on (Y^σ, J_2), the symplectic quotient $Y^\sigma /\!\!/_0 K^\tau$ is Kähler, and the above local diffeomorphism is also Kähler.

2. Since σ preserves $\mu^{-1}(0)$, it descends to an involution $\bar{\sigma}$ on $X/\!\!/\!\!/_0 K$ such that $\bar{\sigma}^* \bar{J}_i = (-1)^i \bar{J}_i$, $\bar{\sigma}^* \bar{\omega}_i = (-1)^i \bar{\omega}_i$ for $i = 1, 2, 3$. The result then follows. □

Following [AB, Hi] and similar constructions in 4-dimensional gauge theory [NR, D1, Pa], we apply the finite dimensional results to the gauge theoretical setup. Recall that P is a principal K-bundle over a compact non-orientable manifold M and let $\tilde{P}$ be the pull-back to the oriented cover $\tilde{M}$. We also assume that $(\tilde{M}, \omega)$ is Kähler with an odd complex dimension and $\tau^*\omega = -\omega$, where τ is the non-trivial deck transformation.

Let $\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^\circ$ be the set of flat connections on $\tilde{P}^{\mathbb{C}}$ such that (i) the stabilizer under the $\mathcal{G}(\tilde{P}^{\mathbb{C}})$ action is $Z(G)$, and (ii) the linearization $\left(\begin{smallmatrix} d_A & -\mathrm{ad}_\psi \\ \mathrm{ad}_\psi & d_A \end{smallmatrix}\right) \colon \Omega^1(M, \mathrm{ad}P)^{\oplus 2} \to \Omega^2(M, \mathrm{ad}P)^{\oplus 2}$ of the curvature is surjective. Notice that when M is a surface, condition (i) implies (ii). In general, $\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^\circ$ is a smooth submanifold in $\mathcal{A}(\tilde{P}^{\mathbb{C}})$, and since the action of $\mathcal{G}(\tilde{P}^{\mathbb{C}})/Z(G)$ on it is free, the subset $\mathcal{M}^{\mathrm{dR}}(\tilde{P}^{\mathbb{C}})^\circ := (\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^\circ \cap \mathcal{A}^{\mathrm{flat,red}}(\tilde{P}^{\mathbb{C}}))/\mathcal{G}(\tilde{P}^{\mathbb{C}})$ is in the smooth part of the moduli space $\mathcal{M}^{\mathrm{dR}}(\tilde{P}^{\mathbb{C}})$ (see for example Goldman [G1] or Kobayashi [Ko, Chapter VII]). The free action of $\mathcal{G}(\tilde{P}^{\mathbb{C}})/Z(G)$ or $\mathcal{G}(\tilde{P})/Z(K)$ from condition (i) implies that 0 is a regular value of $\upmu_J$ on $\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^\circ$, and the subset $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^\circ := \mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^\circ \cap \upmu_J^{-1}(0)/\mathcal{G}(\tilde{P})$ is in the smooth part of Hitchin's moduli space $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})$. By the Donaldson-Corlette theorem, we have the homeomorphism $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^\circ \cong \mathcal{M}^{\mathrm{dR}}(\tilde{P}^{\mathbb{C}})^\circ$.

On the other hand, for the non-orientable manifold M, let $\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})^{\circ} = \{A \in \mathcal{A}(P^{\mathbb{C}}) : \pi^* A \in \mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^{\circ}\}$, $\mathcal{A}^{\mathrm{Hitchin}}(P)^{\circ} = \mathcal{A}^{\mathrm{Hitchin}}(P) \cap \mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})^{\circ}$. Then $\mathcal{M}^{\mathrm{Hitchin}}(P)^{\circ} := \mathcal{A}^{\mathrm{Hitchin}}(P)^{\circ}/\mathcal{G}(P^{\mathbb{C}})$ is in the smooth part of $\mathcal{M}^{\mathrm{Hitchin}}(P)$, but we will not consider here the smooth points of $\mathcal{M}^{\mathrm{Hitchin}}(P)$ that are outside $\mathcal{M}^{\mathrm{Hitchin}}(P)^{\circ}$. By Theorem 2.2 (the analog of the Donaldson-Corlette theorem for non-orientable manifolds), we have a homeomorphism between $\mathcal{M}^{\mathrm{Hitchin}}(P)^{\circ}$ and $\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})^{\circ} := (\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})^{\circ} \cap \mathcal{A}^{\mathrm{flat,red}}(P^{\mathbb{C}}))/\mathcal{G}(P^{\mathbb{C}})$.

We now prove Theorem 1.1.

Proof. 1&3. Note that $\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^{\circ}$ is a τ-invariant Kähler submanifold in $T\mathcal{A}(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$. We now apply Lemma 2.4 to $\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^{\circ}$ on which τ acts preserving ω_J and J. Since τ also acts on $\mathcal{G}(\tilde{P})$ and $\mathcal{G}(P) \cong (\mathcal{G}(\tilde{P}))^{\tau}$, $\mathcal{G}(P)/Z(K)$ acts Hamiltonianly and freely on $\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})^{\circ} \cong (\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^{\circ})^{\tau}$, which is Kähler with respect to J, ω_J. Thus $\mathcal{M}^{\mathrm{Hitchin}}(P)^{\circ} = (\mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})^{\circ} \cap \mu_J^{-1}(0))/\mathcal{G}(P) = \mathcal{A}^{\mathrm{flat}}(P^{\mathbb{C}})^{\circ} /\!\!/_0 \mathcal{G}(P)$ is a symplectic quotient. Since the latter is non-empty, there is a local Kähler diffeomorphism $\mathcal{M}^{\mathrm{Hitchin}}(P)^{\circ} \to (\mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})^{\circ} /\!\!/_0 \mathcal{G}(\tilde{P}))^{\tau} = (\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^{\circ})^{\tau}$.

2. The space $T\mathcal{A}(\tilde{P}) \cong \mathcal{A}(\tilde{P}^{\mathbb{C}})$ with I, J, K is hyper-Kähler and the action of $\mathcal{G}(\tilde{P})$ is Hamiltonian with respect to $\omega_I, \omega_J, \omega_K$. Let $(\mu^{-1}(0))^{\circ}$ be the subset of $\mu^{-1}(0)$ on which $\mathcal{G}(\tilde{P})/Z(K)$ acts freely. Then $\mathcal{M}^{\mathrm{HK}}(\tilde{P})^{\circ} := (\mu^{-1}(0))^{\circ}/\mathcal{G}(\tilde{P})$ is the smooth part of the hyper-Kähler quotient $\mathcal{M}^{\mathrm{HK}}(\tilde{P})$. The involutions τ on $\mathcal{A}(\tilde{P})$ and $\mathcal{G}(\tilde{P})$ satisfy the conditions of Proposition 2.5. So $(\mathcal{M}^{\mathrm{HK}}(\tilde{P})^{\circ})^{\tau}$ is Kähler and totally geodesic with respect to $\bar{J}$ and $\bar{\omega}_J$, and totally real and Lagrangian with respect to $\bar{I}, \bar{K}$ and $\bar{\omega}_I, \bar{\omega}_K$ in $\mathcal{M}^{\mathrm{HK}}(\tilde{P})^{\circ}$. If M is a non-orientable surface, then $\mu_I^{-1}(0) \cap \mu_K^{-1}(0) = \mathcal{A}^{\mathrm{flat}}(\tilde{P}^{\mathbb{C}})$ and $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^{\circ} = \mathcal{M}^{\mathrm{HK}}(\tilde{P})^{\circ}$. In general, $\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^{\circ}$ is a τ-invariant hyper-Kähler submanifold in $\mathcal{M}^{\mathrm{HK}}(\tilde{P})^{\circ}$. The results follow from $(\mathcal{M}^{\mathrm{Hitchin}}(\tilde{P})^{\circ})^{\tau} = \mathcal{M}^{\mathrm{Hitchin}}(\tilde{P}) \cap (\mathcal{M}^{\mathrm{HK}}(\tilde{P})^{\circ})^{\tau}$. □

3. The representation variety perspective

3.1. **Representation variety and Betti moduli space.** Let Γ be a finitely generated group and let G be a connected complex Lie group. Then G acts on $\mathrm{Hom}(\Gamma, G)$ by the conjugate action on G. A representation $\phi \in \mathrm{Hom}(\Gamma, G)$ is *reductive* if the closure of $\phi(\Gamma)$ in G is contained in the Levi subgroup of any parabolic subgroup containing $\phi(\Gamma)$; let $\mathrm{Hom}^{\mathrm{red}}(\Gamma, G)$ be the set of such. The condition $\phi \in \mathrm{Hom}^{\mathrm{red}}(\Gamma, G)$ is equivalent to the statement that the G-orbit $G \cdot \phi$ is closed [GM]. It is also equivalent to the condition that the composition of ϕ with the adjoint representation of G is semi-simple (see [Ri, Section 3] and [Sik, Theorem 30]). The quotient

$$\mathrm{Hom}(\Gamma, G) /\!\!/ G = \mathrm{Hom}^{\mathrm{red}}(\Gamma, G)/G$$

is known as the *representation variety.* A reductive representation $\phi \in \mathrm{Hom}^{\mathrm{red}}(\Gamma, G)$ is *good* [JM] if its stabilizer $G_\phi = Z(G)$; let $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)$ be the set of such. On the other hand, $\phi \in \mathrm{Hom}(\Gamma, G)$ is *Ad-irreducible* if its composition with the adjoint representation of G is an irreducible representation of Γ. Let $\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G)$ be the set of such. Notice that this set is empty unless G is simple. Clearly, $\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G) \subset \mathrm{Hom}^{\mathrm{good}}(\Gamma, G)$. In general, $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)/G$ may not be smooth, but it is so when Γ is the fundamental group of a compact orientable surface [Sik, Corollary 50].

Suppose M is a compact manifold and $P^{\mathbb{C}} \to M$ is a principal G-bundle over M. Choose a base point $x_0 \in M$ and let $\Gamma = \pi_1(M, x_0)$. Then $\mathrm{Hom}(\Gamma, G)/\!\!/G$ is known as the *Betti moduli space* [S2], denoted by $\mathcal{M}^{\mathrm{Betti}}(P^{\mathbb{C}})$. The identification $\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}}) \cong \mathcal{M}^{\mathrm{Betti}}(P^{\mathbb{C}})$, which we recall briefly now, is well known. Given a flat connection, let $T_\alpha \colon P_{\alpha(0)} \to P_{\alpha(1)}$ be the parallel transport along a path α in M. Fix a point $p_0 \in P_{x_0}$ in the fibre over x_0. For $a \in \pi_1(M, x_0)$, choose a loop α based at x_0 representing a, then $\phi(a)$ is the unique element in G defined by $T_\alpha(p_0) = p_0 \phi(a)^{-1}$. If we choose another point in the fibre over x_0, then ϕ differs by a conjugation. Finally, the flat connection is reductive if and only if the corresponding element in $\mathrm{Hom}(\Gamma, G)$ is reductive. Upon identification of the de Rham moduli space $\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})$ and the Betti moduli spaces $\mathcal{M}^{\mathrm{Betti}}(P^{\mathbb{C}}) = \mathrm{Hom}(\Gamma, G)/\!\!/G$, the subset $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)/G$ contains the smooth part $\mathcal{M}^{\mathrm{dR}}(P^{\mathbb{C}})^\circ$ introduced in subsection 2.3; they are equal when M is a compact orientable surface.

If M is non-orientable and $\pi \colon \tilde{M} \to M$ is the oriented cover, we choose a base point $\tilde{x}_0 \in \pi^{-1}(x_0)$ and let $\tilde{\Gamma} = \pi_1(\tilde{M}, \tilde{x}_0)$. Then there is a short exact sequence

$$1 \to \tilde{\Gamma} \to \Gamma \to \mathbb{Z}_2 \to 1$$

and $\tilde{\Gamma}$ can be identified with an index 2 subgroup in Γ. In the rest of this section, we will study the relation of the representation varieties $\mathrm{Hom}(\Gamma, G)/\!\!/G$ and $\mathrm{Hom}(\tilde{\Gamma}, G)/\!\!/G$ or the Betti moduli spaces $\mathcal{M}^{\mathrm{Betti}}(P^{\mathbb{C}})$ and $\mathcal{M}^{\mathrm{Betti}}(\tilde{P}^{\mathbb{C}})$. Some of the results, when M is a compact non-orientable surface, appeared in [Ho], which used different methods.

We first establish a useful fact that was used in subsection 2.2.

Lemma 3.1. *Suppose Γ is a finitely generated group and $\tilde{\Gamma}$ is an index 2 subgroup in Γ. Let G be a connected, complex reductive Lie group. Then $\phi \in \mathrm{Hom}(\Gamma, G)$ is reductive if and only if the restriction $\phi|_{\tilde{\Gamma}} \in \mathrm{Hom}(\tilde{\Gamma}, G)$ is reductive.*

Proof. Recall that $\phi \in \mathrm{Hom}(\Gamma, G)$ is reductive if and only if the composition $\mathrm{Ad} \circ \phi$ is a semisimple representation on $\mathfrak{g}$. Similarly, $\phi|_{\tilde{\Gamma}}$ is reductive if and only if $\mathrm{Ad} \circ \phi|_{\tilde{\Gamma}}$ is semisimple. By $\Gamma/\tilde{\Gamma} \cong \mathbb{Z}_2$ and [Cl], [Bou, Chap. 3, §9.8, Lemme 2], $\mathrm{Ad} \circ \phi$ is semisimple if only if $\mathrm{Ad} \circ \phi|_{\tilde{\Gamma}}$ is so. The result then follows. □

Corollary 3.2. *Let G be a connected, complex reductive Lie group. Suppose P is a principal G-bundle over a compact non-orientable manifold M whose oriented cover is $\pi\colon \tilde{M} \to M$. Then a flat connection A on P is reductive if and only if the pull-back $\pi^* A$ is a flat reductive connection on $\tilde{P} := \pi^* P$.*

3.2. **Representation varieties associated to an index 2 subgroup.** Let $\varGamma$ be a finitely generated group and let $\tilde{\varGamma}$ be an index 2 subgroup in $\varGamma$. Let G be a connected complex Lie group and let $Z(G)$ be its center. For any $c \in \varGamma \setminus \tilde{\varGamma}$, we have $\mathrm{Ad}_c|_{\tilde{\varGamma}} \in \mathrm{Aut}(\tilde{\varGamma})$, and the class $[\mathrm{Ad}_c|_{\tilde{\varGamma}}] \in \mathrm{Aut}(\tilde{\varGamma})/\mathrm{Inn}(\tilde{\varGamma})$ is independent of the choice of c. So we have a homomorphism $\mathbb{Z}_2 \cong \{1, \tau\} \to \mathrm{Aut}(\tilde{\varGamma})/\mathrm{Inn}(\tilde{\varGamma})$ given by $\tau \mapsto [\mathrm{Ad}_c|_{\tilde{\varGamma}}]$.

Lemma 3.3. $\mathbb{Z}_2 \cong \{1, \tau\}$ *acts on* $\mathrm{Hom}(\tilde{\varGamma}, G)/\!\!/G$ *and on* $\mathrm{Hom}^{\mathrm{good}}(\tilde{\varGamma}, G)/G$.

Proof. We define $\tau[\phi] = [\phi \circ \mathrm{Ad}_c]$ for any $\phi \in \mathrm{Hom}(\tilde{\varGamma}, G)$. The action is well-defined since if $[\phi'] = [\phi]$, i.e., $\phi' = \mathrm{Ad}_g \circ \phi$ for some $g \in G$, then $\phi' \circ \mathrm{Ad}_c = \mathrm{Ad}_g \circ \phi \circ \mathrm{Ad}_c \sim \phi \circ \mathrm{Ad}_c$. The τ-action is independent of the choice of c because if $c' \in \varGamma \setminus \tilde{\varGamma}$ is another element, then $c'c^{-1} \in \tilde{\varGamma}$ and $\phi \circ \mathrm{Ad}_{c'} = \mathrm{Ad}_{\phi(c'c^{-1})} \circ (\phi \circ \mathrm{Ad}_c) \sim \phi \circ \mathrm{Ad}_c$. We do have a $\mathbb{Z}_2$-action because $\tau^2[\phi] = [\phi \circ \mathrm{Ad}_{c^2}] = [\mathrm{Ad}_{\phi(c^2)} \circ \phi] = [\phi]$. Finally, if ϕ is in $\mathrm{Hom}^{\mathrm{red}}(\tilde{\varGamma}, G)$ or $\mathrm{Hom}^{\mathrm{good}}(\tilde{\varGamma}, G)$, then so is $\phi \circ \mathrm{Ad}_c$. Thus τ acts on $\mathrm{Hom}(\tilde{\varGamma}, G)/\!\!/G$ and $\mathrm{Hom}^{\mathrm{good}}(\tilde{\varGamma}, G)/G$. $\square$

Proposition 3.4. *There exists a continuous map*

$$L\colon (\mathrm{Hom}^{\mathrm{good}}(\tilde{\varGamma}, G)/G)^{\tau} \to Z(G)/2Z(G). \tag{3.1}$$

So $(\mathrm{Hom}^{\mathrm{good}}(\tilde{\varGamma}, G)/G)^{\tau} = \bigcup_{r \in Z(G)/2Z(G)} \mathcal{N}_r^{\mathrm{good}}$, *where* $\mathcal{N}_r^{\mathrm{good}} := L^{-1}(r)$.

Proof. If $\tau[\phi] = [\phi]$, then there exists $g \in G$ such that $\phi \circ \mathrm{Ad}_c = \mathrm{Ad}_g \circ \phi$. Since $c^2 \in \tilde{\varGamma}$, we have $\mathrm{Ad}_{g^2} \circ \phi = \phi \circ \mathrm{Ad}_{c^2} = \mathrm{Ad}_{\phi(c^2)} \circ \phi$. Thus $z := g^2 \phi(c^2)^{-1} \in G_\phi = Z(G)$. If $[\phi'] = [\phi]$, i.e., $\phi' = \mathrm{Ad}_h \circ \phi$ for some $h \in G$, then $\phi' \circ \mathrm{Ad}_c = \mathrm{Ad}_{g'} \circ \phi'$ for $g' = \mathrm{Ad}_h g$. Since $g'^2 = \mathrm{Ad}_h g^2 = z \mathrm{Ad}_h \phi(c^2) = z \phi'(c^2)$, we obtain $(g')^2 \phi'(c^2)^{-1} = z$.

If $\phi \circ \mathrm{Ad}_{c'} = \mathrm{Ad}_{g'} \circ \phi$ holds for different choices of $c' \in \varGamma \setminus \tilde{\varGamma}$ and $g' \in G$, then $z' = (g')^2 \phi(c'^2)^{-1} \in Z(G)$ from the above discussion. On the other hand, we have $\mathrm{Ad}_{g^{-1}g'} \circ \phi = \mathrm{Ad}_{\phi(c^{-1}c')} \circ \phi$ as $c^{-1}c' \in \tilde{\varGamma}$. Thus $t := (g')^{-1} g \phi(c^{-1}c') \in G_\phi = Z(G)$. We get $t^2 (g')^2 = (tg')^2 = g\phi(c^{-1}c') g \phi(c^{-1}c') = \mathrm{Ad}_g \phi(c^{-1}c') g^2 \phi(c^{-1}c') = \phi(\mathrm{Ad}_c(c^{-1}c')) z \phi(c^2) \phi(c^{-1}c') = \phi((c')^2) z$, i.e., $z' z^{-1} = t^{-2} \in 2Z(G)$. So the map $L\colon [\phi] \mapsto [z] \in Z(G)/2Z(G)$ is well-defined.

Since $\phi \in \mathrm{Hom}^{\mathrm{good}}(\tilde{\varGamma}, G)$, the element $[g] \in G/Z(G)$ is uniquely determined by and depends continuously on ϕ. Therefore $[z] \in Z(G)/2Z(G)$ depends continuously on $[\phi] \in (\mathrm{Hom}^{\mathrm{good}}(\tilde{\varGamma}, G)/G)^{\tau}$. $\square$

If $\phi \in \mathrm{Hom}(\Gamma, G)$ satisfies $\phi|_{\tilde{\Gamma}} \in \mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)$, then $\phi \in \mathrm{Hom}^{\mathrm{good}}(\Gamma, G)$. However, the converse is not necessarily true. Let

$$\mathrm{Hom}^{\mathrm{good}}_{\tau}(\Gamma, G) = \{\phi \in \mathrm{Hom}(\Gamma, G) : \phi|_{\tilde{\Gamma}} \in \mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)\}.$$

We show that if $[\phi] \in (\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)/G)^{\tau}$, then $L([\phi])$ is the obstruction of extending ϕ to a representation of Γ.

Lemma 3.5. *The restriction* $R\colon [\phi] \mapsto [\phi|_{\tilde{\Gamma}}]$ *maps* $\mathrm{Hom}^{\mathrm{good}}_{\tau}(\Gamma, G)/G$ *surjectively to* $\mathcal{N}_0^{\mathrm{good}}$.

Proof. First, $\mathrm{im}(R) \subset \mathcal{N}_0^{\mathrm{good}}$ because for any $\phi \in \mathrm{Hom}^{\mathrm{good}}_{\tau}(\Gamma, G)$, $\phi|_{\tilde{\Gamma}} \in \mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)$, $(\phi|_{\tilde{\Gamma}}) \circ \mathrm{Ad}_c = \mathrm{Ad}_{\phi(c)} \circ \phi|_{\tilde{\Gamma}} \sim \phi|_{\tilde{\Gamma}}$ and $L([\phi|_{\tilde{\Gamma}}]) = [\phi(c)^2\phi(c^2)^{-1}] = 0$. We show that in fact $\mathrm{im}(R) = \mathcal{N}_0^{\mathrm{good}}$. Let $\phi_0 \in \mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)$ such that $\tau[\phi_0] = [\phi_0]$ and $L([\phi_0]) = 0$. Then there exist $g \in G$ and $t \in Z(G)$ such that $\phi_0 \circ \mathrm{Ad}_c = \mathrm{Ad}_g \circ \phi_0$ and $g^2\phi(c^2)^{-1} = t^2$. We can extend ϕ_0 to $\phi \in \mathrm{Hom}(\Gamma, G)$ which is uniquely determined by the requirements $\phi|_{\tilde{\Gamma}} = \phi_0$ and $\phi(c) = gt^{-1}$. Since $\phi_0 \in \mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)$, $\phi \in \mathrm{Hom}^{\mathrm{good}}_{\tau}(\Gamma, G)$ and therefore $[\phi_0] \in \mathrm{im}(R)$. □

Proposition 3.6. $R\colon \mathrm{Hom}^{\mathrm{good}}_{\tau}(\Gamma, G)/G \to \mathcal{N}_0^{\mathrm{good}}$ *is a Galois covering map whose structure group is* $\{s \in Z(G) : s^2 = e\}$.

Proof. We define an action of $\{s \in Z(G) : s^2 = e\}$ on $\mathrm{Hom}^{\mathrm{good}}_{\tau}(\Gamma, G)$. For any such s and $\phi \in \mathrm{Hom}^{\mathrm{good}}_{\tau}(\Gamma, G)$, we define $s{\cdot}\phi$ by $(s{\cdot}\phi)|_{\tilde{\Gamma}} = \phi|_{\tilde{\Gamma}}$ and $(s{\cdot}\phi)|_{\Gamma\setminus\tilde{\Gamma}} = s(\phi|_{\Gamma\setminus\tilde{\Gamma}})$ the group multiplication. It is clear that $s \cdot \phi \in \mathrm{Hom}(\Gamma, G)$. Moreover, since $(s \cdot \phi)|_{\tilde{\Gamma}} = \phi|_{\tilde{\Gamma}} \in \mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)$, $s \cdot \phi \in \mathrm{Hom}^{\mathrm{good}}_{\tau}(\Gamma, G)$. Clearly, the action descends to a well-defined action on $\mathrm{Hom}^{\mathrm{good}}_{\tau}(\Gamma, G)/G$ by $s \cdot [\phi] = [s \cdot \phi]$ preserving the fibres of R.

We show that this action is free. Suppose $s \cdot [\phi] = [\phi]$, then $s \cdot \phi = \mathrm{Ad}_h \circ \phi$ for some $h \in G$. Since $\phi|_{\tilde{\Gamma}} = (s{\cdot}\phi)|_{\tilde{\Gamma}} = \mathrm{Ad}_h \circ (\phi|_{\tilde{\Gamma}}) \in \mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)$, we get $h \in Z(G)$ and hence $s \cdot \phi = \phi$. Then $s\phi(c) = \phi(c)$ implies $s = e$.

It remains to show that the action is transitive on each fibre of R. Let $[\phi], [\phi'] \in \mathrm{Hom}^{\mathrm{good}}_{\tau}(\Gamma, G)$ such that $R([\phi]) = R([\phi'])$. Then there exists an $h \in G$ such that $\phi'|_{\tilde{\Gamma}} = \mathrm{Ad}_h \circ (\phi|_{\tilde{\Gamma}})$. Thus

$$\mathrm{Ad}_{\phi'(c)h}\circ(\phi|_{\tilde{\Gamma}}) = \mathrm{Ad}_{\phi'(c)}\circ(\phi'|_{\tilde{\Gamma}}) = (\phi'|_{\tilde{\Gamma}})\circ\mathrm{Ad}_c = \mathrm{Ad}_h\circ(\phi|_{\tilde{\Gamma}})\circ\mathrm{Ad}_c = \mathrm{Ad}_{h\phi(c)}\circ(\phi|_{\tilde{\Gamma}}).$$

Hence $s := \phi(c)^{-1}h^{-1}\phi'(c)h \in Z(G)$ since $\phi|_{\tilde{\Gamma}} \in \mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)$. Furthermore

$$s^2 = \phi(c)^{-1}sh^{-1}\phi'(c)h = \phi(c^{-2})h^{-1}\phi'(c^2)h = \phi(c^{-2})\phi(c^2) = e.$$

Since $(s{\cdot}\phi)|_{\tilde{\Gamma}} = \phi|_{\tilde{\Gamma}} = \mathrm{Ad}_{h^{-1}}\circ(\phi'|_{\tilde{\Gamma}})$ and $(s{\cdot}\phi)(c) = s\phi(c) = \phi(c)s = (\mathrm{Ad}_{h^{-1}}\circ\phi')(c)$, we get $s \cdot \phi = \mathrm{Ad}_{h^{-1}} \circ \phi'$, or $[\phi'] = [s \cdot \phi]$. □

Corollary 3.7. *Under the above assumptions, there is a local homeomorphism from* $\mathrm{Hom}_\tau^{\mathrm{good}}(\varGamma, G)/G$ *to* $(\mathrm{Hom}^{\mathrm{good}}(\tilde{\varGamma}, G)/G)^\tau$*, which restricts to a local diffeomorphism on the smooth part. If* $|Z(G)|$ *is odd, this local homeomorphism (diffeomorphism, respectively) is a homeomorphism (diffeomorphism, respectively).*

Proof. The first statement follows easily from Propositions 3.4 and 3.6. If $|Z(G)|$ is odd, we get $Z(G)/2Z(G) \cong \{0\}$ and $(\mathrm{Hom}^{\mathrm{good}}(\tilde{\varGamma}, G)/G)^\tau = \mathcal{N}_0^{\mathrm{good}}$ by Proposition 3.4. Furthermore, since $\{s \in Z(G) : s^2 = e\} = \{e\}$, the covering map in Proposition 3.6 is a bijection. □

The involution τ also acts on $\mathrm{Hom}^{\mathrm{irr}}(\tilde{\varGamma}, G)/G$. Let

$$\mathrm{Hom}_\tau^{\mathrm{irr}}(\varGamma, G) = \{\phi \in \mathrm{Hom}(\varGamma, G) : \phi|_{\tilde{\varGamma}} \in \mathrm{Hom}^{\mathrm{irr}}(\tilde{\varGamma}, G)\}.$$

By Propositions 3.4 and 3.6, we get

Corollary 3.8. *If* G *is simple, there exists a decomposition*

$$(\mathrm{Hom}^{\mathrm{irr}}(\tilde{\varGamma}, G)/G)^\tau = \bigcup_{r \in Z(G)/2Z(G)} \mathcal{N}_r^{\mathrm{irr}},$$

where $\mathcal{N}_r^{\mathrm{irr}} = \mathcal{N}_r^{\mathrm{good}} \cap (\mathrm{Hom}^{\mathrm{irr}}(\tilde{\varGamma}, G)/G)^\tau$*. Furthermore, there exists a Galois covering map* $R\colon \mathrm{Hom}_\tau^{\mathrm{irr}}(\varGamma, G)/G \to \mathcal{N}_0^{\mathrm{irr}}$ *with structure group* $\{s \in Z(G) : s^2 = e\}$*. If* $|Z(G)|$ *is odd, then there is a bijection from* $\mathrm{Hom}_\tau^{\mathrm{irr}}(\varGamma, G)/G$ *to* $(\mathrm{Hom}^{\mathrm{irr}}(\tilde{\varGamma}, G)/G)^\tau$*.*

The results in this subsection show parts (1) and (2) of Theorem 1.2.

3.3. **Betti moduli space associated to a non-orientable surface.** By subsection 3.2 or parts (1) and (2) of Theorem 1.2, we know that a representation $\phi \in \mathrm{Hom}^{\mathrm{good}}(\tilde{\varGamma}, G)$ such that $\tau[\phi] = [\phi]$ can be extended to one on $\varGamma$ if an only if $L([\phi]) = 0$. When applied to $\varGamma = \pi_1(M)$ and $\tilde{\varGamma} = \pi_1(\tilde{M})$, where M is non-orientable and $\tilde{M}$ is its oriented cover, we conclude that a τ-invariant flat bundle over the $\tilde{M}$ corresponding to $\phi \in \mathrm{Hom}^{\mathrm{good}}(\tilde{\varGamma}, G)$ is the pull-back of a flat bundle over M if and only if $L([\phi]) = 0$. We now consider the example when $M = \varSigma$ is a compact non-orientable surface, in which case we can characterize all the components $\mathcal{N}_r^{\mathrm{good}}$ explicitly. The principal G-bundles on $\varSigma$ are topologically classified by $H^2(\varSigma, \pi_1(G)) \cong \pi_1(G)/2\pi_1(G)$ whereas those on the oriented cover $\tilde{\varSigma}$ are classified by $H^2(\tilde{\varSigma}, \pi_1(G)) \cong \pi_1(G)$. The classes in these groups are the obstructions of lifting the structure group G of the bundles to its universal cover group.

A compact non-orientable surface $\varSigma$ is of the form $\varSigma_k^\ell$ ($\ell \geq 0$, $k = 1, 2$), the connected sum of $2\ell + k$ copies of $\mathbb{R}P^2$. Then $\tilde{\varSigma}$ is a compact surface of genus $2\ell + k - 1$. For $k = 1$, we have

$$\pi_1(\varSigma) = \left\langle\, a_i, b_i\, (1 \leq i \leq \ell), c : c^{-2} \textstyle\prod_{i=1}^{\ell} [a_i, b_i] \,\right\rangle,$$

$$\pi_1(\tilde{\varSigma}) = \left\langle\, a_i, b_i, a_i', b_i'\, (1 \leq i \leq \ell) : \textstyle\prod_{i=1}^{\ell} [a_i, b_i] \prod_{i=1}^{\ell} [a_i', b_i'] \,\right\rangle.$$

The inclusion $\pi_1(\tilde{\Sigma}) \to \pi_1(\Sigma)$ is given by $a_i \mapsto a_i$, $b_i \mapsto b_i$, $a'_i \mapsto \mathrm{Ad}_c b_i$, $b'_i \mapsto \mathrm{Ad}_c a_i$ $(1 \le i \le \ell)$. For $k = 2$, we have

$$\pi_1(\Sigma) = \big\langle\, a_i, b_i\,(1 \le i \le \ell), c, d : d^{-1}cd^{-1}c^{-1} \textstyle\prod_{i=1}^{\ell}[a_i, b_i] \,\big\rangle,$$

$$\pi_1(\tilde{\Sigma}) = \big\langle a_0, b_0, a_i, b_i, a'_i, b'_i\,(1 \le i \le \ell) : [a_0, b_0] \textstyle\prod_{i=1}^{\ell}[a_i, b_i] \prod_{i=1}^{\ell}[a'_i, b'_i] \,\big\rangle.$$

The inclusion $\pi_1(\tilde{\Sigma}) \to \pi_1(\Sigma)$ is given by $a_0 \mapsto d^{-1}$, $b_0 \mapsto c^2$, $a_i \mapsto a_i$, $b_i \mapsto b_i$, $a'_i \mapsto \mathrm{Ad}_{d^{-1}c} b_i$, $b'_i \mapsto \mathrm{Ad}_{d^{-1}c} a_i$ $(1 \le i \le \ell)$. In both cases, $c \in \pi_1(\Sigma) \setminus \pi_1(\tilde{\Sigma})$.

While a flat G-bundle over Σ may be non-trivial, its pull-back to $\tilde{\Sigma}$ is always trivial topologically. We assume that G is semi-simple, simply connected and denote $PG = G/Z(G)$. Then $\pi_1(PG) = Z(G)$ and $H^2(\Sigma, \pi_1(PG)) \cong Z(G)/2Z(G)$. The map $O\colon \mathrm{Hom}(\pi_1(\Sigma), PG)/PG \to Z(G)/2Z(G)$ that gives the obstruction class can be explicitly described as follows [HL1]. Let $\phi \in \mathrm{Hom}(\pi_1(\Sigma), PG)$. For $k = 1$, let $\widetilde{\phi(a_i)}, \widetilde{\phi(b_i)}, \widetilde{\phi(c)} \in G$ be any lifts of $\phi(a_i), \phi(b_i), \phi(c) \in PG$, respectively. Then $O([\phi])$ is the element in $Z(G)/2Z(G)$ represented by $\widetilde{\phi(c)}^2 (\prod_{i=1}^{\ell}[\widetilde{\phi(a_i)}, \widetilde{\phi(b_i)}])^{-1} \in Z(G)$. (It is easy to check that the class in $Z(G)/2Z(G)$ is independent of the lifts.) The description of the case $k = 2$ is similar. Consequently, there is a decomposition $\mathrm{Hom}(\pi_1(\Sigma), PG)/PG = \bigcup_{r \in Z(G)/2Z(G)} \mathcal{M}_r$, where $\mathcal{M}_r = O^{-1}(r)$.

Let $G \to PG$, $g \mapsto \bar{g}$ be the quotient map. The induced map $\mathrm{Hom}(\pi_1(\Sigma), G) \to \mathrm{Hom}(\pi_1(\Sigma), PG)$ is denoted by $\phi \mapsto \bar{\phi}$. In this section, we need to be restricted to Ad-irreducible representations. The reason is that ϕ is Ad-irreducible if and only if $\bar{\phi}$ is so, whereas if ϕ is good, $\bar{\phi}$ is not necessarily so and its stabilizer may be larger than $Z(G)$. We have $\mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG = \bigcup_{r \in Z(G)/2Z(G)} \mathcal{M}_r^{\mathrm{irr}}$, where $\mathcal{M}_r^{\mathrm{irr}} = \mathcal{M}_r \cap (\mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG)$.

Lemma 3.9. *There is a natural map*

$$\Psi\colon (\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), G)/G)^\tau \to \mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG$$

satisfying $L = O \circ \Psi$. *Consequently,* Ψ *maps* $\mathcal{N}_r^{\mathrm{irr}}$ *to* $\mathcal{M}_r^{\mathrm{irr}}$ *for each* $r \in Z(G)/2Z(G)$.

Proof. Given $[\phi] \in (\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), G)/G)^\tau$, there exists $g \in G$ (which is unique up to $Z(G)$ since $G_\phi = Z(G)$) such that $\mathrm{Ad}_g \circ \phi = \phi \circ \mathrm{Ad}_c$. We define $\check{\phi} \in \mathrm{Hom}(\pi_1(\Sigma), PG)$ by $\check{\phi}|_{\pi_1(\tilde{\Sigma})} = \bar{\phi}$ and $\check{\phi}(c) = \bar{g}$. The representation $\check{\phi}$ is a homomorphism because $\check{\phi}(c)^2 = \bar{g}^2 = \bar{\phi}(c^2)$, which follows from $z = g^2\phi(c^2)^{-1} \in Z(G)$ in Proposition 3.4. Since $\bar{\phi} \in \mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), PG)$, we have $\check{\phi} \in \mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)$. We define Ψ by $\Psi([\phi]) = [\check{\phi}]$. To show that $O([\check{\phi}]) = L([\phi]) = [z]$, we work in the case $k = 1$. By using the respective lifts $\phi(a_i), \phi(b_i), g \in G$ of $\check{\phi}(a_i), \check{\phi}(b_i), \check{\phi}(c) \in PG$, we get $O([\check{\phi}]) = [g^2(\prod_{i=1}^{\ell}[\phi(a_i), \phi(b_i)])^{-1}] = [g^2\phi(c^2)^{-1}] = [z]$, where we have used the relation $\prod_{i=1}^{\ell}[\phi(a_i), \phi(b_i)] = c^2$ in $\pi_1(\tilde{\Sigma})$. The case $k = 2$ is similar. $\square$

Proposition 3.10. *The map* $\Psi\colon (\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde{\Sigma}), G)/G)^\tau \to \mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma), PG)/PG$ *is surjective. Consequently,* $\Psi\colon \mathcal{N}_r^{\mathrm{irr}} \to \mathcal{M}_r^{\mathrm{irr}}$ *is surjective for each* $r \in Z(G)/2Z(G)$.

Proof. Let $[\phi]\in\mathrm{Hom}_\tau^{\mathrm{irr}}(\Sigma,PG)/PG$. Although $\phi(c)\in PG$, $\mathrm{Ad}_{\phi(c)}$ acts on G. We show the case $k=1$ only. Fix the lifts $\widetilde{\phi(a_i)},\widetilde{\phi(b_i)}\in G$ of $\phi(a_i),\phi(b_i)\in PG$. Define $\tilde\phi\in\mathrm{Hom}(\pi_1(\tilde\Sigma),G)$ by $\tilde\phi(a_i)=\widetilde{\phi(a_i)}$, $\tilde\phi(b_i)=\widetilde{\phi(b_i)}$, $\tilde\phi(a_i')=\mathrm{Ad}_{\phi(c)}\tilde\phi(b_i)$, $\tilde\phi(b_i')=\mathrm{Ad}_{\phi(c)}\tilde\phi(a_i)$, for $i=1,\dots,\ell$. This indeed defines a representation because

$$\prod_{i=1}^{\ell}[\tilde\phi(a_i),\tilde\phi(b_i)]\prod_{i=1}^{\ell}[\tilde\phi(a_i'),\tilde\phi(b_i')]=\prod_{i=1}^{\ell}[\tilde\phi(a_i),\tilde\phi(b_i)]\,\mathrm{Ad}_{\phi(c)}\Big(\prod_{i=1}^{\ell}[\tilde\phi(a_i),\tilde\phi(b_i)]\Big)^{-1}=e.$$

The last equality is because $\prod_{i=1}^{\ell}[\tilde\phi(a_i),\tilde\phi(b_i)]\in G$ projects to $\phi(c)^2\in PG$. Since ϕ is Ad-irreducible, so is $\tilde\phi$. $[\tilde\phi]$ is τ-invariant because $\tilde\phi\circ\mathrm{Ad}_c=\mathrm{Ad}_{\phi(c)}\circ\tilde\phi$, which can be checked on the generators: $\tilde\phi(\mathrm{Ad}_c a_i)=\tilde\phi(b_i')=\mathrm{Ad}_{\phi(c)}\tilde\phi(a_i)$, $\tilde\phi(\mathrm{Ad}_c a_i')=\mathrm{Ad}_{\phi(c^2)}\tilde\phi(b_i)=\mathrm{Ad}_{\phi(c)}\tilde\phi(a_i')$, etc. It is then obvious that $\Psi([\tilde\phi])=[\phi]$. □

For the group PG, since $Z(PG)$ is trivial, $(\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde\Sigma),PG)/PG)^\tau$ does not decompose according to Proposition 3.4 and the map $\bar R\colon\mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma),PG)/PG\to(\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde\Sigma),PG)/PG)^\tau$ in Proposition 3.6 is bijective. The map Ψ is in fact the composition of $(\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde\Sigma),G)/G)^\tau\to(\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde\Sigma),PG)/PG)^\tau$ (induced by $G\to PG$) followed by $\bar R^{-1}$. So for each $r\in Z(G)/2Z(G)$, the component $\mathcal N_r^{\mathrm{irr}}$ of the fixed point set $(\mathrm{Hom}^{\mathrm{irr}}(\pi_1(\tilde\Sigma),G)/G)^\tau$ corresponds precisely to the component $\mathcal M_r^{\mathrm{irr}}$ of $\mathrm{Hom}_\tau^{\mathrm{irr}}(\pi_1(\Sigma),PG)/PG$ which consists of flat PG-bundles over Σ of topological type $r\in Z(G)/2Z(G)$. In particular, $\mathcal N_0^{\mathrm{irr}}$ corresponds to the component $\mathcal M_0^{\mathrm{irr}}$ of topologically trivial flat PG-bundles over Σ.

The results in subsection shows part (3) of Theorem 1.2.

4. Comparison of representation variety and gauge theoretical constructions

Suppose M is a compact non-orientable manifold, $\pi\colon\tilde M\to M$ is the oriented cover, and $\tau\colon\tilde M\to\tilde M$ is the non-trivial deck transformation. In subsection 2.2, we considered the natural lift of τ on $\tilde P^{\mathbb C}=\pi^*P^{\mathbb C}$, where $P^{\mathbb C}$ is a principal G-bundle over M. Such a lift, still denoted by τ, is a G-bundle map satisfying $\tau^2=\mathrm{id}_{\tilde P^{\mathbb C}}$ and induces involutions on the space $\mathcal A(\tilde P^{\mathbb C})$ of connections on $\tilde P^{\mathbb C}$ and various moduli spaces. Moduli spaces associated to $P^{\mathbb C}\to M$ are then related to the τ-invariant parts of those associated to $\tilde P^{\mathbb C}\to\tilde M$ (cf. Theorem 1.1, especially part 3). This can also be seen in the language of representation varieties (cf. Lemma 3.5, Proposition 3.6 on $\mathcal N_0^{\mathrm{good}}$ and Corollary 3.7). To provide a geometric interpretation of the rest of the results in subsections 3.2 and 3.3 on $\mathcal N_r^{\mathrm{good}}$ or $\mathcal N_r^{\mathrm{irr}}$ when $r\neq0$, we will need to generalize the setting in gauge theory.

Suppose $Q\to\tilde M$ is a principal G-bundle and the non-trivial deck transformation τ on $\tilde M$ is lifted to a bundle map τ_Q on Q, which is not necessarily an involution. Let A be an irreducible connection on Q that is invariant under τ_Q up to a gauge transformation, i.e., $\tau_Q^*A=\varphi^*A$ for $\varphi\in\mathcal G(Q)$. Since $(\tau_Q\circ\varphi^{-1})^2$ is a gauge transformation on Q which fixes A, it is in the center $Z(G)$. So by modifying τ_Q with a gauge transformation φ, we can assume that τ_Q satisfies $\tau_Q^2 = z \in Z(G)$. In this way, although τ_Q is not strictly an involution, it is so up to a gauge transformation, the right action of z on Q. Since φ and hence τ_Q can be adjusted by an element in $Z(G)$, $z = \tau_Q^2$ is well defined modulo $2Z(G)$. If $z = t^2 \in 2Z(G)$ ($t \in Z(G)$), then z can be absorbed in τ_Q by a redefinition such that τ_Q is an honest involution, and we are back to the situation before. In the general case when $\tau_Q^2 = z \in Z(G)$ is not the identity element, since $Z(G)$ acts trivially on the connections as gauge transformations, the action $\tau_Q^*\colon \mathcal{A}(Q) \to \mathcal{A}(Q)$ of τ_Q on connections is still an honest involution. So we can define the invariant subspace $\mathcal{A}(Q)^{\tau_Q}$ and much of the analysis in subsections 2.2 and 2.3 applies.

We now consider flat connections and relate this generalized setting to our results on representation varieties. Choose base points $x_0 \in M$ and $\tilde{x}_0 \in \pi^{-1}(x_0) \subset \tilde{M}$, and let $\mathit{\Gamma} = \pi_1(M, x_0)$, $\tilde{\mathit{\Gamma}} = \pi_1(\tilde{M}, \tilde{x}_0)$. We fix an element $c \in \mathit{\Gamma} \setminus \tilde{\mathit{\Gamma}}$.

Proposition 4.1. *For any $z \in Z(G)$, there is a 1-1 correspondence between the following two sets:*
(1) isomorphism classes of pairs (Q, A), where $Q \to \tilde{M}$ is a principal G-bundle with a G-bundle map τ_Q lifting the deck transformation τ on $\tilde{M}$ satisfying $\tau_Q^2 = z$, A is a τ_Q-invariant flat connection on Q
and
(2) equivalence classes of pairs (ϕ, g) under the diagonal adjoint action of G, where $\phi \in \mathrm{Hom}(\tilde{\mathit{\Gamma}}, G)$ and $g \in G$ satisfy $\phi \circ \mathrm{Ad}_c = \mathrm{Ad}_g \circ \phi$ and $g^2\phi(c^2)^{-1} = z$.

Proof. Given a bundle Q and a τ_Q-invariant flat connection A, let $T_\alpha\colon Q_{\alpha(0)} \to Q_{\alpha(1)}$ be the parallel transport along a path $\alpha\colon [0,1] \to \tilde{M}$. τ_Q-invariance of the connection implies $\tau_Q \circ T_\alpha = T_{\tau\circ\alpha} \circ \tau_Q$ for any path α. Let γ be a path in $\tilde{M}$ from $\tilde{x}_0$ to $\tau(\tilde{x}_0)$ so that $[\pi \circ \gamma] = c$. Choose $q_0 \in Q_{\tilde{x}_0}$ and let $g \in G$ be defined by $T_\gamma q_0 = \tau_Q(q_0)g^{-1}$. On the other hand, define $\phi \in \mathrm{Hom}(\tilde{\mathit{\Gamma}}, G)$ by $T_\alpha q_0 = q_0\phi(a)^{-1}$ for any $a \in \tilde{\mathit{\Gamma}}$, where α is a loop in $\tilde{M}$ based at $\tilde{x}_0$ such that $[\alpha] = a$. To check the conditions on (ϕ, g), we note that $\tau_Q(T_\alpha q_0) = \tau_Q(q_0)\phi(a)^{-1}$ and $T_{\tau\circ\alpha}\tau_Q(q_0) = T_\gamma \circ T_{\gamma\cdot(\tau\circ\alpha)\cdot\gamma^{-1}}(q_0 g) = (T_\gamma q_0)\phi(\mathrm{Ad}_c a)g = \tau_Q(q_0)\mathrm{Ad}_g^{-1}\phi(\mathrm{Ad}_c a)$. So τ_Q-invariance implies $\phi(\mathrm{Ad}_c a) = \mathrm{Ad}_g\phi(a)$ for all $a \in \tilde{\mathit{\Gamma}}$. Similar calculations give $\tau_Q(T_\gamma q_0) = \tau_Q(\tau_Q(q_0)g^{-1}) = q_0 z g^{-1}$ and $T_{\tau\circ\gamma}(\tau_Q q_0) = T_{\gamma\cdot(\tau\circ\gamma)}(q_0 g) = q_0\phi(c^2)^{-1}g$ which imply $g^2\phi(c^2)^{-1} = z$. If another point $q_0' = q_0 h \in Q_{\tilde{x}_0}$ is chosen (where $h \in G$), then the resulting pair is $(\phi', g') = (\mathrm{Ad}_{h^{-1}} \circ \phi, \mathrm{Ad}_{h^{-1}} g)$.

Conversely, given a pair (ϕ, g) satisfying the conditions, we want to construct a bundle Q together with a lifting τ_Q of τ such that $\tau_Q^2 = z$ and a τ_Q-invariant flat connection on Q. Let $\hat{M}$ be the universal covering space of $\tilde{M}$ (and of M). Then $\tilde{\mathit{\Gamma}}$ and $\mathit{\Gamma}$ act on $\hat{M}$, and $\tilde{M} = \hat{M}/\tilde{\mathit{\Gamma}}$, $M = \hat{M}/\mathit{\Gamma}$. Let $Q = \hat{M} \times_{\tilde{\mathit{\Gamma}}} G$, that is, points in Q are equivalence classes $[(x, h)]$, where $x \in \hat{M}$ and $h \in G$, and $(xa, h) \sim (x, \phi(a)h)$ for any $a \in \tilde{\Gamma}$. Let $\tau_Q\colon Q \to Q$ be defined by $\tau_Q\colon [(x,h)] \mapsto [(xc^{-1}, gh)]$. To check that τ_Q is well-defined, we note that for any $a \in \tilde{\Gamma}$, $(xac^{-1}, gh) \sim (xc^{-1}, \phi(\mathrm{Ad}_c a)gh) = (xc^{-1}, g\phi(a)h)$. Clearly, τ_Q commutes with the right G-action on Q. Furthermore, $\tau_Q^2 = z$ because $\tau_Q^2\colon [(x,h)] \mapsto [(xc^{-2}, g^2h)] = [(x, \phi(c^{-2})g^2h)] = [(x,h)]z$. It is easy to see that the trivial connection on $\hat{M} \times G$ is $\tilde{\Gamma}$-invariant and descends to a flat connection on Q. The latter is invariant under τ_Q since the trivial connection on $\hat{M} \times G$ is invariant under $(x,h) \mapsto (xc^{-1}, gh)$. Moreover, this connection induces the pair (ϕ, g). □

Remark 4.2. We explain the gauge theoretic perspective of the results in subsections 3.2 and 3.3 using the correspondence in Proposition 4.1.

1. As we noted, the τ is lifted to a G-bundle map τ_Q on $Q \to \tilde{M}$ such that $\tau_Q^2 = z \in Z(G)$, then z is detemined up to $2Z(G)$. Likewise, $z = g^2\phi(c^2)^{-1}$ is determined also modulo $2Z(G)$ by $[\phi] \in (\mathrm{Hom}^{\mathrm{good}}(\tilde{\Gamma}, G)/G)^\tau$ (Proposition 3.4). If $\tau_Q^2 = t^2$ for some $t \in Z(G)$, then τ_Q can be redefined as $\tau'_Q = \tau_Q t^{-1}$ so that $(\tau'_Q)^2 = \mathrm{id}_Q$. We then have a G-bundle $Q/\tau'_Q \to M$ over the non-orientable manifold M whose pull-back of to $\tilde{M}$ is Q. If a flat connection is invariant under τ_Q, it is also invariant under τ'_Q and hence descends to a flat connection on Q/τ'_Q. This is the situation in Lemma 3.5 and Proposition 3.6 (where Q/τ'_Q was $P^{\mathbb{C}}$). In fact, from these results, we see that $[z] \in Z(G)/2Z(G)$ is the obstruction to the existence of a flat G-bundle on M whose pull-back to $\tilde{M}$ is Q.
2. In general, $\tau_Q^2 \neq \mathrm{id}_Q$ and the quotient of Q by the subgroup generated by τ_Q is a bundle over M with a fibre smaller than G. However, the PG-bundle $\bar{Q} := Q/Z(G)$ over $\tilde{M}$ does have an honest involution $\tau_{\bar{Q}}$. So $\bar{Q}$ descends to a PG-bundle $\bar{Q}/\tau_{\bar{Q}}$ over M. Moreover, a τ_Q-invariant flat connection on Q descends to a $\tau_{\bar{Q}}$-invariant flat connection on $\bar{Q}$ and hence to a flat PG-connection on $\bar{Q}/\tau_{\bar{Q}}$. The bundle $\bar{Q}/\tau_{\bar{Q}} \to M$ is usually non-trivial as its structure group can not be lifted to G. (Otherwise, Q would be its pull-back to $\tilde{M}$ and would admit a lift τ_Q of τ so that $\tau_Q^2 = \mathrm{id}_Q$.) Proposition 3.10 shows that when G is simply connected and when $M = \Sigma$ is a non-orientable surface, the topological type, i.e., the obstruction to lifting the PG-bundle $\bar{Q}/\tau_{\bar{Q}}$ to a G-bundle over M is precisely $[z] \in Z(G)/2Z(G)$.

Remark 4.3. 1. We can use $\tilde{x}_1 = \tau(\tilde{x}_0)$ as an another base point of the fundamental group of $\tilde{M}$ so that $\tilde{x}_0$ and $\tilde{x}_1$ play symmetric roles. The image of $\pi_1(\tilde{\Sigma}, \tilde{x}_1)$ under π_* can be identified with $\tilde{\Gamma} \subset \Gamma$. The isomorphism $\tau_*\colon \tilde{\Gamma} \to \pi_1(\tilde{\Sigma}, \tilde{x}_1) \cong \tilde{\Gamma}$ is then $a \mapsto \mathrm{Ad}_c^{-1} a$. Having chosen $q_0 \in Q_{\tilde{x}_0}$, let $q_1 = \tau_Q(q_0) \in Q_{\tilde{x}_1}$ and define $\phi_1\colon \pi_1(\tilde{\Sigma}, \tilde{x}_1) \to G$ by $T_\alpha q_1 = q_1\phi_1([\alpha])^{-1}$, where α is a loop in $\tilde{\Sigma}$ based at $\tilde{x}_1$. Using the identity $\tau_Q \circ T_{\tau\circ\alpha} = T_\alpha \circ \tau_Q$, we obtain $\phi_1([\alpha]) = \phi([\tau\circ\alpha])$. Since $\tau_Q^2 = z$, we also have the identity $T_\gamma z = \tau_Q \circ T_{\tau\circ\gamma} \circ \tau_Q$. So upon the identification of $Q_{\tilde{x}_0}$ and $Q_{\tilde{x}_1}$ by τ_Q, the parallel transports along γ and $\tau\circ\gamma$ differ by z.
2. When $M = \Sigma$ is a non-orientable surface, the approach of double base points was taken in [Ho, HL2]. Consider for example the case $M = \Sigma_1^\ell$. Let α_i, β_i $(1 \le i \le \ell)$

be loops in the oriented cover $\tilde{\Sigma}$ based at $\tilde{x}_0$ and let γ be a path in from $\tilde{x}_0$ to $\tilde{x}_1$ so that $[\pi \circ \alpha_i] = a_i$, $[\pi \circ \beta_i] = b_i$, $[\pi \circ \gamma] = c$. Then an element in $\mathcal{N}_r$ ($r = [z] \in Z(G)/2Z(G)$) can be represented by $(A_i, B_i, C; A_i', B_i', C') \in G^{4\ell+2}$ satisfying $A_i' = A_i$, $B_i' = B_i$, $C' = Cz$, where $A_i, B_i, C, A_i', B_i', C'$ are the holonomies along the loops or paths $\alpha_i, \beta_i, \gamma, \tau \circ \alpha_i, \tau \circ \beta_i, \tau \circ \gamma$ ($1 \le i \le \ell$), respectively. By the above discussion, we have the pattern $A_i = \phi([\alpha_i]) = \phi_1([\tau \circ \alpha_i]) = A_i'$, $B_i = \phi([\beta_i]) = \phi_1([\tau \circ \beta_i]) = B_i'$ ($1 \le i \le \ell$), $C' = Cz$ as in [Ho, HL2].

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