Conditions of smoothness of moduli spaces of flat connections and of representation varieties

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Abstract. We use gauge theoretic and algebraic methods to reexamine the sufficient conditions for smooth points on the moduli space of flat connections on a compact manifold and on the representation variety of a finitely generated and presented group. In particular, we give a complete proof of the slice theorem for the action of the group of gauge transformations on the space of flat connections. We show that on a compact non-orientable surface, flat connections with minimal isotropy need not descend to smooth points on the moduli space.
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1. Introduction

Let G be a reductive complex Lie group. The moduli space of flat G-connections on a manifold M is the quotient of the space of flat reductive connections on principal G-bundles over M modulo the group of gauge transformations. It is well known that this moduli space can be identified with the representation variety of the fundamental group $\pi_1(M)$, i.e., the set of reductive homomorphisms from $\pi_1(M)$ to G modulo conjugations by G.

The motivation for using gauge theory is that one can compute information about the deformation space in terms of the topology and geometry of the bundle and the underlying manifold. For example, on a Riemann surface the dimension of the moduli space follows easily from the Riemann-Roch theorem (see for example [1]). If the manifold is compact and Kähler, then Goldman and Millson [12] were able to show that the singularities in the moduli space were quadratic. In subsequent work, Simpson [34] used the theory of Higgs bundles to place restrictions on the fundamental group of a compact Kähler manifold.

On the other hand, one can study representation varieties very concretely. If M is compact, then $\pi_1(M)$ is finitely generated and finitely presented, and the representation variety is the quotient of an affine variety by G acting on it algebraically. Goldman [11], and later Goldman and Millson [12], expressed deformations in terms of group cohomology of $\pi_1(M)$. This can be traced back to the work of Weil [35] on deformations of discrete subgroups of Lie groups and of Nijenhuis and Richardson [29, 30] on deformations of graded Lie algebras and of homomorphisms of Lie groups and Lie algebras. A common ingredient in these studies is cohomology of groups and algebras.

In this paper, we reexamine deformations in moduli spaces and in representation varieties using the above two approaches, with an emphasis on finding sufficient conditions for smooth points on these spaces.

In §2, we study the local model and smoothness of the moduli space of flat connections from the gauge theory construction. Let P be a principal G-bundle over a compact manifold M. We give a complete proof of the slice theorem for the action on the space of flat connections on P by the group $\mathcal{G}(P)$ of gauge transformations. If a flat connection D is good (i.e., reductive and having stabiliser $Z(G)$), then a neighbourhood of the gauge equivalence class $[D]$ in the moduli space is homeomorphic to a neighbourhood of D in the slice. If in addition $H^2(M, \operatorname{ad} P) = H^2(M, Z(\mathfrak{g}))$, where $\mathfrak{g}$ is the Lie algebra of G, then the moduli space is smooth at $[D]$. Attaining these results involves manipulation of infinite dimensional spaces which has occurred in a number of contexts, for example, in the moduli of (anti-)self-dual connections on four manifolds [2, 6], where the structure group G was assumed to be a compact Lie group. However, in our situation, G is complex reductive, and the action of $\mathcal{G}(P)$ on the space of connections or the space of sections is not an isometry. Instead, we adapt the proofs in [20, 21] for the moduli of holomorphic bundles and of Hermitian-Einstein connections on Kähler manifolds to the study of moduli of flat G-connections on Riemannian manifolds. We include the proofs as we can not find them in the existing literature. Finally, we compare the moduli spaces of flat connections on a non-orientable manifold and of those on its orientation double cover.

In §3, we study the same problems for the representation variety of any finitely generated and finitely presented group Π, such as the fundamental group of a compact manifold. The space $\operatorname{Hom}(\Pi, G)$ of homomorphisms from Π to G is an affine variety in the product of finite copies of G defined by the relations among the generators of Π, and G acts on it algebraically. Therefore we can apply Luna's slice theorem [23] to obtain similar results as in the gauge theoretic approach. Note that $H^2(\Pi, \mathfrak{g}_{\operatorname{Ad}\circ\phi})$ always contains $H^2(\Pi, Z(\mathfrak{g}))$ and $H^2(\Pi, \mathfrak{g}_{\operatorname{Ad}\circ\phi}) = H^2(\Pi, Z(\mathfrak{g}))$ is the condition

under which any infinitesimal deformation from ϕ can be integrated [11]. However, using the implicit function theorem, we found that $\phi \in \mathrm{Hom}(\Pi, G)$ is a smooth point on $\mathrm{Hom}(\Pi, G)$ if the quantity $\dim \mathrm{Hom}(\bar{N}, \mathfrak{g}_{\mathrm{Ad} \circ \phi})^{\Pi} - \dim H^2(\Pi, \mathfrak{g}_{\mathrm{Ad} \circ \phi})$ reaches its maximum at ϕ. Here $\bar{N}$ is the relation module of the presentation of Π and $\mathfrak{g}_{\mathrm{Ad} \circ \phi}$ is the Lie algebra $\mathfrak{g}$ on which Π acts via $\mathrm{Ad} \circ \phi$. We verify the same result using Fox calculus and its appearance in the free resolution of $\mathbb{Z}$ by $\mathbb{Z}\Pi$-modules. If there is a single relation among the generators of Π or if there are no relations among the relators, then the above condition reduces to the minimality of $\dim H^2(\Pi, \mathfrak{g}_{\mathrm{Ad} \circ \phi})$. This is the case, for example, when Π is the fundamental group of a compact (orientable or non-orientable) surface. In this case, the moduli space of flat G-connections is smooth at $[\phi]$ if first, ϕ is reductive and its stabiliser is $Z(G)$ (which implies $H^0(\Pi, \mathfrak{g}_{\mathrm{Ad} \circ \phi}) = Z(\mathfrak{g}))$ and second, $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad} \circ \phi}) = H^2(\Pi, Z(\mathfrak{g}))$. If the surface M is orientable, then $H^0(\Pi, \mathfrak{g}_{\mathrm{Ad} \circ \phi}) \cong H^2(\Pi, \mathfrak{g}_{\mathrm{Ad} \circ \phi})$ by Poincaré duality, and hence the conditions $H^0(\Pi, \mathfrak{g}_{\mathrm{Ad} \circ \phi}) = Z(\mathfrak{g})$ and $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad} \circ \phi}) = Z(\mathfrak{g})$ are equivalent [11]. But if M is non-orientable, the two cohomology groups $H^0(\Pi, \mathfrak{g}_{\mathrm{Ad} \circ \phi})$ and $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad} \circ \phi})$ are different and we have two separate conditions for smoothness; we find an explicit formula for $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad} \circ \phi})$ using Fox calculus. We give an example to show that, in contrast to the case of a compact orientable surface, points in $\mathrm{Hom}(\Pi, G)$ with minimal isotropy may not project to smooth points on the representation variety.

In §4, we compare the gauge theoretic approach to moduli spaces in §2 and the algebraic approach to representation varieties in §3. Using a spectral sequence, we find a relation between $H^2(M, \mathrm{ad}\, P)$ and $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad} \circ \phi})$, where P is a flat G-bundle over M, $\Pi = \pi_1(M)$, and $\phi \in \mathrm{Hom}(\Pi, G)$ is determined by the holonomy of P. Using it, we find that the condition $H^2(M, \mathrm{ad}\, P) = H^2(M, Z(\mathfrak{g}))$ is stronger than $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad} \circ \phi}) = H^2(\Pi, Z(\mathfrak{g}))$ in general. However, if M is a compact (orientable or non-orientable) surface, then $H^2(M, \mathrm{ad}\, P) = H^2(\Pi, \mathfrak{g}_{\mathrm{Ad} \circ \phi})$, and the sufficient conditions of smoothness from the two approaches are identical.

In the Appendix, we present some examples of flat G-connections on orientable and non-orientable manifolds (or homomorphisms from the fundamental groups into G) that we referred to in the main text.

All classical complex Lie groups are reductive, and the complexification of a compact Lie group is also complex reductive (cf. [31, Theorem 3, p.234]). Though we assumed that G is a complex reductive Lie group throughout, our methods clearly apply (with the exception of the use of Luna's slice theorem in §3.3, which is not necessary when G is compact) and our results also hold when G is a compact Lie group .

2. Gauge theory approach to moduli spaces

2.1. **Preliminaries.** Let M be a compact manifold of dimension n, G be a complex reductive group with Lie algebra $\mathfrak{g}$, and $P \to M$ be a principal G-bundle. Let $\mathcal{A}(P)$ be the set of G-connections on P and $\mathcal{A}^{\mathrm{flat}}(P)$ be the subset of flat connections. The group $\mathcal{G}(P)$ of gauge transformations acts on $\mathcal{A}(P)$ preserving $\mathcal{A}^{\mathrm{flat}}(P)$. The centre $Z(G)$ of G can be identified with the subgroup of $\mathcal{G}(P)$ of constant gauge transformations and it acts trivially on $\mathcal{A}(P)$.

A connection D on P induces a connection $D^{\mathrm{ad}\, P}$ on the adjoint bundle $\mathrm{ad}\, P := P \times_{\mathrm{Ad}} \mathfrak{g}$ and defines the covariant differentials $D_i = D_i^{\mathrm{ad}\, P} : \Omega^i(M, \mathrm{ad}\, P) \to \Omega^{i+1}(M, \mathrm{ad}\, P)$, where $0 \le i < \dim M$. If D is flat, then

$$(2.1) \qquad 0 \to \Omega^0(M, \mathrm{ad}\, P) \xrightarrow{D_0} \Omega^1(M, \mathrm{ad}\, P) \xrightarrow{D_1} \cdots \xrightarrow{D_{n-1}} \Omega^n(M, \mathrm{ad}\, P) \to 0$$

is an elliptic complex. Let $H^i(M, \mathrm{ad}\, P) := \ker D_i / \mathrm{im}\, D_{i-1}$ be the de Rham cohomology groups with coefficients in $\mathrm{ad}\, P$. Since M is compact, the *Betti numbers* $b_i(M, \mathrm{ad}\, P) := \dim_{\mathbb{C}} H^i(M, \mathrm{ad}\, P)$ are finite. Let $\chi(M, \mathrm{ad}\, P) := \sum_{i=0}^{n} (-1)^i b_i(M, \mathrm{ad}\, P)$ be the *Euler characteristic* of the complex (2.1). In fact, $\chi(M, \mathrm{ad}\, P) = \chi(M) \dim_{\mathbb{C}} G$.

The Lie algebra $\mathfrak{g}$ of a reductive Lie group G has a decomposition $\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{g}'$, where $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$. Let Ad', ad' denote the restrictions to $\mathfrak{g}'$ of the adjoint representations of G and $\mathfrak{g}$, respectively. Then there is also a decomposition $\mathrm{ad}\, P = Z(\mathfrak{g})_M \oplus \mathrm{ad}'\, P$, where $Z(\mathfrak{g})_M$ is the trivial bundle over M with fiber $Z(\mathfrak{g})$ and $\mathrm{ad}'\, P := P \times_{\mathrm{Ad}'} \mathfrak{g}'$. The decomposition is preserved by the connection D, which is trivial on $Z(\mathfrak{g})_M$ and will be denoted by the same notation on $\mathrm{ad}'\, P$. We have

$$\Omega^i(M, \mathrm{ad}\, P) = \Omega^i(M, Z(\mathfrak{g})) \oplus \Omega^i(M, \mathrm{ad}'\, P), \quad H^i(M, \mathrm{ad}\, P) = H^i(M, Z(\mathfrak{g})) \oplus H^i(M, \mathrm{ad}'\, P),$$

$$b_i(M, \mathrm{ad}\, P) = b_i(M) \dim_{\mathbb{C}} Z(\mathfrak{g}) + b_i(M, \mathrm{ad}'\, P), \quad \chi(M, \mathrm{ad}'\, P) = \chi(M) \dim_{\mathbb{C}}(G/Z(G)).$$

There is a G-invariant symmetric non-degenerate complex bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$ such that the decomposition $\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{g}'$ is orthogonal. It defines a fiberwise complex bilinear form on $\mathrm{ad}\, P$ preserved by the connection and such that the splitting $\mathrm{ad}\, P = Z(\mathfrak{g})_M \oplus \mathrm{ad}'\, P$ is also orthogonal.

So far, the compact manifold M can be either orientable or non-orientable. Now we assume that M is orientable. Using the above complex bilinear form of $\operatorname{ad} P$ and an orientation on M, there is a non-degenerate complex bilinear pairing $\langle \cdot, \cdot \rangle$ on $\Omega^\bullet(M, \operatorname{ad} P)$ given by

$$(2.2) \qquad \Omega^i(M, \operatorname{ad} P) \times \Omega^{n-i}(M, \operatorname{ad} P) \to \mathbb{C}, \qquad (\alpha, \beta) \mapsto \langle \alpha, \beta \rangle := \int_M (\alpha, \wedge \beta).$$

This induces the following Poincaré duality on the de Rham cohomology with coefficients in a flat bundle.

Lemma 2.1. *If M is orientable, there is a non-degenerate complex bilinear pairing, still denoted by $\langle \cdot, \cdot \rangle$,*

$$(2.3) \qquad H^i(M, \operatorname{ad} P) \times H^{n-i}(M, \operatorname{ad} P) \to \mathbb{C}.$$

Hence there is an isomorphism $H^i(M, \operatorname{ad} P)^* \cong H^{n-i}(M, \operatorname{ad} P)$, and $b_i(M, \operatorname{ad} P) = b_{n-i}(M, \operatorname{ad} P)$. The results remain true if the bundle $\operatorname{ad} P$ is replaced by $\operatorname{ad}' P$.

Proof. Consider the dual complex of (2.1),

$$0 \leftarrow \Omega^0(M, \operatorname{ad} P)' \xleftarrow{D_0^\mathsf{T}} \Omega^1(M, \operatorname{ad} P)' \xleftarrow{D_1^\mathsf{T}} \cdots \xleftarrow{D_{n-1}^\mathsf{T}} \Omega^n(M, \operatorname{ad} P)' \leftarrow 0,$$

where $\Omega^i(M, \operatorname{ad} P)'$ is the space of $(n-i)$-currents on M with coefficients in $\operatorname{ad} P$ and D_i^T is the transpose of D_i. Let $H_i(\Omega^\bullet(M, \operatorname{ad} P)') = \ker D_{i-1}^\mathsf{T} / \operatorname{im} D_i^\mathsf{T}$. Then (2.2) induces a non-degenerate pairing

$$H^i(M, \operatorname{ad} P) \times H_i(\Omega^\bullet(M, \operatorname{ad} P)') \to \mathbb{C}.$$

The non-degenerate pairing (2.2) also defines an inclusion $\Omega^{n-i}(M, \operatorname{ad} P) \hookrightarrow \Omega^i(M, \operatorname{ad} P)'$, and the restriction of the map D_{i-1}^T, whose domain is $\Omega^i(M, \operatorname{ad} P)'$, to $\Omega^{n-i}(M, \operatorname{ad} P)$ is $(-1)^{i-1} D_{n-i}$. By a slight generalisation (to allow coefficients in $\operatorname{ad} P$) of [5, Thm 14, p.79], the above (co)homology groups defined using currents can also be computed by smooth forms, i.e., $H_i(\Omega^\bullet(M, \operatorname{ad} P)') \cong H^{n-i}(M, \operatorname{ad} P)$. The results on $\operatorname{ad} P$ then follow. Finally, the pairing (2.2) splits block-diagonally under the decomposition $\Omega^1(M, \operatorname{ad} P) = \Omega^1(M, Z(\mathfrak{g})) \oplus \Omega^i(M, \operatorname{ad}' P)$, and the results on $\operatorname{ad}' P$ also follow. $\qquad \square$

2.2. The slice theorem.

A flat connection $D \in \mathcal{A}^{\text{flat}}(P)$ is *simple* if the kernel of $D_0 \colon \Omega^0(M, \operatorname{ad}' P) \to \Omega^1(M, \operatorname{ad}' P)$ is zero, or if $H^0(M, \operatorname{ad}' P) = 0$. In this case the stabiliser of a simple flat connection has minimal dimension. We choose a Riemannian metric on M and a Hermitian structure on $\operatorname{ad} P$ such that $\operatorname{ad}' P$ is orthogonal to $Z(\mathfrak{g})_M$. Then there are Hermitian inner products on $\Omega^i(M, \operatorname{ad} P)$, $\Omega^i(M, \operatorname{ad}' P)$ regardless of whether M is orientable or not. Let $D_i^\dagger$ be the formal adjoint of D_i. For $D \in \mathcal{A}^{\text{flat}}(P)$, the *slice* at D is the subset

$$(2.4) \qquad \mathcal{S}(D) := \left\{ \alpha \in \Omega^1(M, \operatorname{ad} P) : D_1 \alpha + \tfrac{1}{2}[\alpha, \alpha] = 0, \ D_0^\dagger \alpha = 0 \right\}.$$

The first goal is to prove a slice theorem for an arbitrary principal bundle with a reductive structure group. Let $\Omega_k^\bullet(M, \operatorname{ad} P)$, $\mathcal{A}_k^{\text{flat}}(P)$, $\mathcal{G}_k(P)$, $\mathcal{S}_k(D)$ be the respective spaces completed according to the Sobolev norm $\| \cdot \|_{2,k}$.

Theorem 2.2. *Let M be a compact manifold, G be a complex reductive Lie group and $P \to M$ be a G-bundle. Fix $k > \frac{n}{2}$. Suppose $D \in \mathcal{A}_k^{\text{flat}}(P)$ is simple. Then there is a neighborhood $\mathcal{V}$ of $0 \in \mathcal{S}_k(D)$ such that the map*

$$\mathcal{G}_{k+1}(P)/Z(G) \times \mathcal{V} \to \mathcal{A}_k^{\text{flat}}(P), \qquad ([g], \alpha) \mapsto g \cdot (D + \alpha)$$

is a homeomorphism onto its image.

Proof. We follow in part the proof of slice theorem in [20, 21], though the context there was on holomorphic structure on vector bundles. Let $(\ker D_0)_{k+1}^\perp$ be the orthogonal complement of $\ker D_0$ in $\Omega_{k+1}^0(M, \operatorname{ad} P)$. Consider the map

$$F \colon (\ker D_0)_{k+1}^\perp \times \Omega_k^1(M, \operatorname{ad} P) \to \Omega_k^0(M, \operatorname{ad} P), \qquad (u, \alpha) \mapsto D_0^\dagger(e^u \cdot (D + \alpha) - D).$$

The differential of F along u is $-D_0^\dagger D_0$, which is an invertible operator on $(\ker D_0)_{k+1}^\perp$. By the implicit function theorem, if α is in a sufficiently small neighbourhood $\mathcal{V}$, there exists $u \in (\ker D_0)_{k+1}^\perp$ such that $\beta := e^u \cdot (D + \alpha) - D$ satisfies $D_0^\dagger \beta = 0$. Since D is simple, there is a small neighbourhood $\mathcal{U}$ of $0 \in (\ker D_0)_{k+1}^\perp$ such that the map $u \in \mathcal{U} \mapsto [e^u] \in \mathcal{G}_{k+1}(P)/Z(G)$ is a diffeomorphism onto its image. In this way, we obtain the desired homeomorphism provided we restrict to small (close to identity) gauge transformations. To get the full version, suppose $D + \alpha_1, D + \alpha_2$ are flat connections such that α_1, α_2 are small. We want to show that if $g \in \mathcal{G}_{k+1}(P)$ such that $g \cdot (D + \alpha_1) = D + \alpha_2$, then $[g] \in \mathcal{G}_{k+1}(P)/Z(G)$ is also small, i.e., g is close to $Z(G)$. If $\mathfrak{g}'$ is a simple Lie algebra, then the simple connection D on P induces a simple connection on $\operatorname{ad}' P$, i.e., the kernel of $D_0^{\operatorname{End}(\operatorname{ad}' P)} \colon \Omega^0(M, \operatorname{End}(\operatorname{ad}' P)) \to \Omega^1(M, \operatorname{End}(\operatorname{ad}' P))$ is zero. We apply the arguments in [20, 21] to the vector bundle $\operatorname{ad}' P$. Since $D^{\operatorname{ad}' P} + \operatorname{ad}'(\alpha_1), D^{\operatorname{ad}' P} + \operatorname{ad}'(\alpha_2)$ are

connections on $\mathrm{ad}'\,P$ related by the gauge transformation $\mathrm{Ad}'(g)$, we have $\mathrm{Ad}'(g) = c(\mathrm{id}_{\mathrm{ad}'\,P} + g')$ for some constant $c \neq 0$ and a small $g' \in \Gamma_{k+1}(\mathrm{End}(\mathrm{ad}'\,P))$. More precisely, we have

$$\|g'\|_{2,k+1} \leq \frac{c_2 \|\mathrm{id}_{\mathrm{ad}'\,P}\|_{2,k+1}(\|\alpha_1\|_{2,k} + \|\alpha_2\|_{2,k})}{c_1 - c_2(\|\alpha_1\|_{2,k} + \|\alpha_2\|_{2,k})},$$

where $c_1, c_2 > 0$ are constants such that $\|[D^{\mathrm{ad}'\,P}, g']\|_{2,k} \geq c_1 \|g'\|_{2,k+1}$ and $\|g'\alpha\|_{2,k} \leq c_2 \|g'\|_{2,k+1}\|\alpha\|_{2,k}$, respectively. We want to show that c is close to 1, and hence $\mathrm{Ad}'(g)$ is close to $\mathrm{id}_{\mathrm{ad}'\,P}$, or g is close to $Z(G)$. Let $c_3 > 0$ be a constant such that $\|[\xi, \eta]\|_{2,k} \leq c_3 \|\xi\|_{2,k}\|\eta\|_{2,k}$ for all $\xi, \eta \in \Omega^0_k(M, \mathrm{ad}'\,P)$. Since $\mathfrak{g}'$ is non-Abelian, we can fix ξ, η such that $\zeta := [\xi, \eta] \in \Omega^0(M, \mathrm{ad}'\,P)$ is non-zero. Let $c_4 = \|\zeta\|_{2,k}$, $c_5 = c_2 c_3 \|\xi\|_{2,k}\|\eta\|_{2,k}$ and $c_6 = c_2 c_5$ be positive constants. Then we have

$$\|[\xi, g'\eta]\|_{2,k} \text{ and } \|[g'\xi, \eta]\|_{2,k} \leq c_5 \|g'\|_{2,k+1}, \qquad \|[g'\xi, g'\eta]\|_{2,k} \leq c_6 \|g'\|_{2,k+1}^2.$$

Finally $\mathrm{Ad}'(g)\zeta = [\mathrm{Ad}'(g)\xi, \mathrm{Ad}'(g)\eta]$, or $\zeta + g'\zeta = c\,[\xi + g'\xi, \eta + g'\eta]$ implies

$$\frac{c_4(1 - c_2\|g'\|_{2,k+1})}{c_4 + 2c_5\|g'\|_{2,k+1} + c_6\|g'\|_{2,k+1}^2} \leq c \leq \frac{c_4(1 + c_2\|g'\|_{2,k+1})}{c_4 - 2c_5\|g'\|_{2,k+1} - c_6\|g'\|_{2,k+1}^2}.$$

When $\|\alpha_1\|_{2,k}$, $\|\alpha_2\|_{2,k}$ are sufficiently small, so is $\|g'\|_{2,k+1}$, and thus c is sufficiently close to 1. Therefore $\mathrm{Ad}'(g)$ is close to the identity. In general, $\mathfrak{g}'$ is semisimple and suppose $\mathfrak{g}' = \oplus_{i=1}^r \mathfrak{g}_i$ is a decomposition into simple Lie algebras. Let Ad'_i be the restriction of the adjoint representation of G to $\mathfrak{g}_i$. Then there is a decomposition of vector bundles $\mathrm{ad}'\,P = \oplus_{i=1}^r \mathrm{ad}'_i P$, where $\mathrm{ad}'_i P := P \times_{\mathrm{ad}'_i} \mathfrak{g}_i$, which is respected by both the gauge transformation $\mathrm{Ad}'(g)$ and the induced connection on $\mathrm{ad}'\,P$. The above argument shows that on each $\mathrm{ad}'_i P$, $\mathrm{Ad}'_i(g)$ is close to the identity. Therefore so is the whole $\mathrm{Ad}'(g)$ and hence the result. $\qquad\square$

In the above proof, if $\alpha \in \Omega^1(M, \mathrm{ad}\,P)$ is smooth, then so is the solution $u \in (\ker D_0)_{k+1}^{\perp}$ to the elliptic equation $F(u, \alpha) = 0$. Similarly, if $\alpha_1, \alpha_2 \in \Omega^1(M, \mathrm{ad}\,P)$ are smooth, then so is the solution $g \in \mathcal{G}(P)$ to the elliptic equation $g^{-1}Dg = \alpha_1 - \mathrm{Ad}(g^{-1})\alpha_2$. Therefore the result of Theorem 2.2 remains true if the Sobolev spaces are replaced by spaces $\Omega^{\bullet}(M, \mathrm{ad}\,P)$, $\mathcal{A}(P)$, $\mathcal{G}(P)$, $\mathcal{S}(D)$ of smooth objects.

A flat connection D is *reductive* if the closure of the holonomy group $\mathrm{Hol}(D)$ is contained in the Levi subgroup of any parabolic subgroup containing $\mathrm{Hol}(D)$. Equivalently, D is reductive if its orbit under $\mathcal{G}(P)$ is closed. Let $\mathcal{A}^{\mathrm{flat,red}}(P)$ be the set of flat, reductive connections on P. The *moduli space of flat connections* on P is $\mathcal{M}^{\mathrm{flat}}(P) := \mathcal{A}^{\mathrm{flat,red}}(P)/\mathcal{G}(P)$. If a flat connection D is simple, then its stabiliser $\mathcal{G}(P)_D$ contains $Z(G)$ as a subgroup of finite index. A flat connection D is *good* (cf. [18]) if it is reductive and its stabiliser $\mathcal{G}(P)_D$ is $Z(G)$. The slice at a good connection gives a local model for the moduli space.

Corollary 2.3. *Let D be a reductive, simple flat connection on P. Then the map*

$$p \colon \mathcal{S}(D)/\mathcal{G}(P)_D \to \mathcal{A}^{\mathrm{flat}}(P)/\mathcal{G}(P), \qquad [\alpha] \mapsto [D + \alpha]$$

is a homeomorphism of a neighbourhood of $[0]$ in $\mathcal{S}(D)/\mathcal{G}(P)_D$ onto a neighbourhood of $[D]$ in $\mathcal{M}^{\mathrm{flat}}(P)$. If in addition D is good, then the homeomorphism is from a neighbourhood of 0 in $\mathcal{S}(D)$ onto a neighbourhood of $[D]$ in $\mathcal{M}^{\mathrm{flat}}(P)$.

Proof. By Theorem 2.2, p is well defined, continuous and one-to-one near $[0]$. For a sufficiently small neighbourhood $\mathcal{U}$ of 0 in $\mathcal{S}(D)$ such that the connection $D + \alpha$ is reductive and simple for any $\alpha \in \mathcal{U}$, the preimage of $p(\mathcal{U}/\mathcal{G}(P)) \subset \mathcal{M}^{\mathrm{flat}}(P)$ in $\mathcal{A}^{\mathrm{flat}}(P)$ under the projection is $\mathcal{G}(P) \cdot \mathcal{U}$. This is an open subset and, since the orbit $\mathcal{G}(P) \cdot D$ is closed, it descends to an open neighbourhood of $[D]$ in $\mathcal{M}^{\mathrm{flat}}(P)$. So p^{-1} is also continuous. $\qquad\square$

2.3. **Smooth points on the moduli space.** A separate issue is the smoothness of $\mathcal{A}^{\mathrm{flat}}(P)$ itself near a flat connection D. We will show that $H^2(M, \mathrm{ad}'\,P)$ is the *obstruction* to smoothness from the gauge theoretic point of view. The proof outlined below is similar to the proofs in [20, 21] for moduli of holomorphic bundles and of Hermitian-Einstein connections on Kähler manifolds. However we are studying moduli of flat G-connections on a Riemannian manifold. With the Riemannian metric on M and the Hermitian structure chosen above, the Laplacian $\square_i := D_i^{\dagger}D_i + D_{i-1}D_{i-1}^{\dagger}$ and the associated Green's operator G_i preserve the decomposition $\Omega^i(M, \mathrm{ad}\,P) \cong \Omega^i(M, Z(\mathfrak{g})) \oplus \Omega^i(M, \mathrm{ad}'\,P)$ for $0 \leq i \leq n$. In particular, a harmonic form in $\Omega^i(M, \mathrm{ad}\,P)$ projects to harmonic forms in $\Omega^i(M) \otimes Z(\mathfrak{g})$ and $\Omega^i(M, \mathrm{ad}'\,P)$. We define the *Kuranishi map*

$$\kappa \colon \Omega^1(M, \mathrm{ad}\,P) \to \Omega^1(M, \mathrm{ad}\,P), \qquad \alpha \mapsto \alpha + \tfrac{1}{2}D_1^{\dagger}G_2[\alpha, \alpha].$$

Proposition 2.4. *Let M be a compact manifold and G be a complex reductive Lie group. Suppose D is a flat connection on a G-bundle $P \to M$ such that $H^2(M, \mathrm{ad}'\,P) = 0$. Then for $k > \frac{n}{2} + 2$, there is a neighbourhood of D in $\mathcal{A}_k^{\mathrm{flat}}(P)$ and a neighbourhood of 0 in $\ker D_1 \subset \Omega_k^1(M, \mathrm{ad}\,P)$ that are diffeomorphic via the map $D + \alpha \mapsto \kappa(\alpha)$.*

Proof. As the differential of κ is the identity map at 0, it is invertible near 0. For $\alpha \in \Omega^1(M, \operatorname{ad} P)$, we have $[\alpha, \alpha] \in \Omega^2(M, \operatorname{ad}' P)$. By the assumption $H^2(M, \operatorname{ad}' P) = 0$, the Green's operator G_2 is the inverse of the Laplacian $\Box_2 = D_2^\dagger D_2 + D_1 D_1^\dagger$ on $\Omega^2(M, \operatorname{ad}' P)$. Therefore $\Box_2 G_2[\alpha, \alpha] = G_2 \Box_2[\alpha, \alpha] = [\alpha, \alpha]$ and

$$D_1 \kappa(\alpha) = D_1 \alpha + \tfrac{1}{2} D_1 D_1^\dagger G_2[\alpha, \alpha] = D_1 \alpha + \tfrac{1}{2}[\alpha, \alpha] - \tfrac{1}{2} G_2 D_2^\dagger D_2[\alpha, \alpha].$$

Here since $\Box_2$ commutes with $D_1 D_1^\dagger$ and $D_2^\dagger D_2$, so does G_2. If $D_1 \alpha + \tfrac{1}{2}[\alpha, \alpha] = 0$, then $-\tfrac{1}{2} D_2[\alpha, \alpha] = D_2 D_1 \alpha = 0$, and so $D_1 \kappa(\alpha) = 0$. Conversely, if $D_1 \kappa(\alpha) = 0$, we want to show that $\gamma := D_1 \alpha + \tfrac{1}{2}[\alpha, \alpha]$ is zero. Using

$$\gamma = \tfrac{1}{2} G_2 D_2^\dagger D_2[\alpha, \alpha] = G_2 D_2^\dagger[D_1 \alpha, \alpha] = G_2 D_2^\dagger[\gamma, \alpha],$$

we get, for some constant $c_0 > 0$,

$$\|\gamma\|_{2,k-1} \le c_0 \|\alpha\|_{2,k-2} \|\gamma\|_{2,k-2} \le c_0 \|\alpha\|_{2,k} \|\gamma\|_{2,k-1}.$$

Therefore $\gamma = 0$ when $\|\alpha\|_{2,k}$ is sufficiently small. $\square$

From the proof of Proposition 2.4, it is evident that the result holds if we assume, instead of $H^2(M, \operatorname{ad}' P) = 0$, that the harmonic part of $[\alpha, \alpha]$ is zero for all $\alpha \in \Omega^1(M, \operatorname{ad} P)$. The condition itself depends on a metric on M, but it implies that the map $H^1(M, \operatorname{ad} P) \to H^2(M, \operatorname{ad}' P)$, sending the cohomology class represented by a closed 1-form $\alpha \in H^1(M, \operatorname{ad} P)$ to the cohomology class of $[\alpha, \alpha]$, is zero.

As in the slice theorem, Proposition 2.4 remains valid if we restrict to the spaces of smooth objects. Let $\mathcal{A}_\circ^{\mathrm{flat}}(P)$ be the set of reductive flat connections that are good and satisfy the condition $H^2(M, \operatorname{ad}' P) = 0$, and let $\mathcal{M}_\circ^{\mathrm{flat}}(P) := \mathcal{A}_\circ^{\mathrm{flat}}(P)/\mathcal{G}(P)$. Combining Proposition 2.4 and Corollary 2.3, we conclude that $\mathcal{M}_\circ^{\mathrm{flat}}(P)$ is in the smooth part of the moduli space $\mathcal{M}^{\mathrm{flat}}(P)$.

Corollary 2.5. *Suppose $D \in \mathcal{A}_\circ^{\mathrm{flat}}(P)$. Then there is a neighbourhood of $[D]$ in $\mathcal{M}_\circ^{\mathrm{flat}}(P)$ diffeomorphic to a neighbourhood of 0 in $H^1(M, \operatorname{ad} P)$. In particular, $\dim_{\mathbb{C}} \mathcal{M}_\circ^{\mathrm{flat}}(P) = b_1(M, \operatorname{ad} P)$ is finite.*

When M is a compact orientable surface (i.e., $n = 2$), the two cohomology groups $H^0(M, \operatorname{ad}' P)$ and $H^2(M, \operatorname{ad}' P)$ are dual spaces of each other by Lemma 2.1. So a good flat connection represents a smooth point in the moduli space. Moreover, the dimension of $\mathcal{M}_\circ^{\mathrm{flat}}(P)$ can be computed by an index formula.

Corollary 2.6. *Suppose M is a compact orientable surface of genus $g > 1$ and $P \to M$ is a flat principal G-bundle. Then for any good flat $D \in \mathcal{A}^{\mathrm{flat}}(P)$, we have $H^2(M, \operatorname{ad}' P) = 0$, and there is a neighbourhood of $[D]$ in $\mathcal{M}_\circ^{\mathrm{flat}}(P)$ diffeomorphic to a neighbourhood of 0 in $H^1(M, \operatorname{ad} P)$. Moreover,*

$$\dim_{\mathbb{C}} \mathcal{M}_\circ^{\mathrm{flat}}(P) = (2g - 2) \dim_{\mathbb{C}} G + 2 \dim_{\mathbb{C}} Z(\mathfrak{g}).$$

Proof. By Corollary 2.5, the dimension of the moduli space is

$$b_1(M, \operatorname{ad} P) = b_1(M, \operatorname{ad}' P) + b_1(M) \dim_{\mathbb{C}} Z(\mathfrak{g}) = -\chi(M, \operatorname{ad}' P) + 2g \dim_{\mathbb{C}} Z(\mathfrak{g})$$
$$= (2g - 2) \dim_{\mathbb{C}} \mathfrak{g}' + 2g \dim_{\mathbb{C}} Z(\mathfrak{g}) = (2g - 2) \dim_{\mathbb{C}} \mathfrak{g} + 2 \dim_{\mathbb{C}} Z(\mathfrak{g}). \qquad \square$$

In this case ($n = 2$), the pairing (2.3) is a (complex) symplectic structure on $H^1(M, \operatorname{ad} P)$. In fact, the symplectic reduction procedure of [1] yields a holomorphic symplectic form ω on $\mathcal{M}_\circ^{\mathrm{flat}}(P)$ which restricts to the above one on the tangent space at each $[D] \in \mathcal{M}_\circ^{\mathrm{flat}}(P)$. More generally, if $\dim M = n$ is even and M is a compact Kähler manifold with a Kähler form ω, then a symplectic form ω on $\mathcal{M}_\circ^{\mathrm{flat}}(P)$, or $\omega_{[D]}$ on $H^1(M, \operatorname{ad} P)$, is defined by

$$(2.5) \qquad \omega_{[D]}([\alpha], [\beta]) = \int_M (\alpha, \wedge \beta) \wedge \frac{\omega^{\frac{n}{2}-1}}{(\frac{n}{2}-1)!},$$

where $\alpha, \beta \in \Omega^1(M, \operatorname{ad} P)$ are closed.

2.4. **Non-orientable manifolds.** Now suppose M is a compact, non-orientable manifold of dimension n. Let $\pi \colon \tilde{M} \to M$ be the orientable double cover. The non-trivial deck transformation $\tau \colon \tilde{M} \to \tilde{M}$ is an involution on $\tilde{M}$. Given a principal bundle $P \to M$ with complex reductive structure group G, let $\tilde{P} := \pi^* P$ denote the pullback to $\tilde{M}$. Then there is a lift of the involution τ to $\tilde{P}$, still denoted by τ, such that $\tilde{P}/\tau = P$, and the map π^* pulls back forms, connections and gauge transformation from M to $\tilde{M}$. The pullback map τ^* acts as an involution on $\Omega^i(\tilde{M}, \operatorname{ad} \tilde{P})$, $\mathcal{A}(\tilde{P})$, $\mathcal{G}(\tilde{P})$, and the τ^*-invariant subspaces can be identified with the corresponding spaces from the bundle $P \to M$ [14], i.e., we have the following isomorphisms via π^*:

$$\Omega^i(M, \operatorname{ad} P) \cong \Omega^i(\tilde{M}, \operatorname{ad} \tilde{P})^\tau, \quad \mathcal{A}(P) \cong \mathcal{A}(\tilde{P})^\tau, \quad \mathcal{G}(P) \cong \mathcal{G}(\tilde{P})^\tau.$$

Suppose a connection D on P pulls back to $\tilde{D}$ on $\tilde{P}$. Then $\tilde{D}$ is flat if and only if D is so, and $\mathcal{A}^{\mathrm{flat}}(P) \cong \mathcal{A}^{\mathrm{flat}}(\tilde{P})^\tau$. The covariant differentials $D_i \colon \Omega^i(M, \operatorname{ad} P) \to \Omega^{i+1}(M, \operatorname{ad} P)$ and $\tilde{D}_i \colon \Omega^i(\tilde{M}, \operatorname{ad} \tilde{P}) \to \Omega^{i+1}(\tilde{M}, \operatorname{ad} \tilde{P})$ satisfy

$$\pi^* \circ D_i = \tilde{D}_i \circ \pi^*, \qquad \tau^* \circ \tilde{D}_i = \tilde{D}_i \circ \tau^*.$$

Therefore τ^* acts as an involution on $H^i(\tilde{M}, \operatorname{ad}\tilde{P})$ for $0 \leq i \leq n$. Let

$$H^i(\tilde{M}, \operatorname{ad}\tilde{P}) = H^i(\tilde{M}, \operatorname{ad}\tilde{P})^\tau \oplus H^i(\tilde{M}, \operatorname{ad}\tilde{P})^{-\tau}$$

be the decomposition such that $\tau^* = \pm 1$ on the subspaces $H^i(\tilde{M}, \operatorname{ad}\tilde{P})^{\pm\tau}$, respectively. Set $b_i^\pm(\tilde{M}, \operatorname{ad}\tilde{P}) := \dim_{\mathbb{C}} H^i(\tilde{M}, \operatorname{ad}\tilde{P})^{\pm\tau}$. We have similar decompositions for $H^i(\tilde{M}, \mathbb{C})$, $H^i(\tilde{M}, \operatorname{ad}'\tilde{P})$, and $b_i^\pm(\tilde{M}) = \dim_{\mathbb{C}} H^i(\tilde{M}, \mathbb{C})^{\pm\tau}$, $b_i^\pm(\tilde{M}, \operatorname{ad}'\tilde{P}) = \dim_{\mathbb{C}} H^i(\tilde{M}, \operatorname{ad}'\tilde{P})^{\pm\tau}$.

Lemma 2.7. *Suppose a flat connection D on $P \to M$ and it lifts to a flat connection $\tilde{D}$ on $\tilde{P} \to \tilde{M}$.*
1. There are non-degenerate bilinear pairings

$$(2.6) \qquad\qquad H^i(\tilde{M}, \operatorname{ad}\tilde{P})^{\pm\tau} \times H^{n-i}(\tilde{M}, \operatorname{ad}\tilde{P})^{\mp\tau} \to \mathbb{C}.$$

Hence $(H^i(\tilde{M}, \operatorname{ad}\tilde{P})^{\pm\tau})^* \cong H^{n-i}(\tilde{M}, \operatorname{ad}\tilde{P})^{\mp\tau}$, $b_i^\pm(\tilde{M}, \operatorname{ad}\tilde{P}) = b_{n-i}^\mp(\tilde{M}, \operatorname{ad}\tilde{P})$.
2. There are isomorphisms $H^i(\tilde{M}, \operatorname{ad}\tilde{P})^\tau \cong H^i(M, \operatorname{ad}P)$, $H^i(\tilde{M}, \operatorname{ad}\tilde{P})^{-\tau} \cong H^{n-i}(M, \operatorname{ad}P)^*$. Hence

$$b_i^+(\tilde{M}, \operatorname{ad}\tilde{P}) = b_i(M, \operatorname{ad}P), \quad b_i^-(\tilde{M}, \operatorname{ad}\tilde{P}) = b_{n-i}(M, \operatorname{ad}P), \quad b_i(\tilde{M}, \operatorname{ad}\tilde{P}) = b_i(M, \operatorname{ad}P) + b_{n-i}(M, \operatorname{ad}P).$$

The same results hold for $H^i(\tilde{M}, \mathbb{C})$ and $H^i(\tilde{M}, \operatorname{ad}'\tilde{P})$.

Proof. 1. Since τ reverses the orientation on $\tilde{M}$, we have, for all $[\alpha] \in H^i(\tilde{M}, \operatorname{ad}\tilde{P})$, $[\beta] \in H^{n-i}(\tilde{M}, \operatorname{ad}\tilde{P})$,

$$\langle \tau^*[\alpha], \tau^*[\beta] \rangle = -\langle [\alpha], [\beta] \rangle.$$

Therefore, the non-degenerate pairing $\langle \cdot, \cdot \rangle$ for $H^\bullet(\tilde{M}, \operatorname{ad}\tilde{P})$ splits into two in (2.6).
2. The first isomorphism is because of the isomorphism $\pi^*\colon \Omega^\bullet(M, \operatorname{ad}P) \to \Omega^\bullet(\tilde{M}, \operatorname{ad}\tilde{P})^\tau$ of cochain complexes. Then $H^i(\tilde{M}, \operatorname{ad}\tilde{P})^{-\tau} \cong (H^{n-i}(\tilde{M}, \operatorname{ad}\tilde{P})^\tau)^* \cong H^{n-i}(M, \operatorname{ad}P)^*$. The rest follows easily. $\qquad\square$

Since $\chi(\tilde{M}, \operatorname{ad}\tilde{P}) = \chi(\tilde{M})\dim_{\mathbb{C}}G$, $\chi(M, \operatorname{ad}P) = \chi(M)\dim_{\mathbb{C}}G$ and $\chi(M) = \frac{1}{2}\chi(\tilde{M})$, we get $\chi(M, \operatorname{ad}P) = \frac{1}{2}\chi(\tilde{M}, \operatorname{ad}\tilde{P})$; both sides vanish if $\dim M = n$ is odd. On the other hand, the Lefschetz number of τ is the supertrace of τ^* on $H^\bullet(\tilde{M}, \operatorname{ad}\tilde{P})$, i.e.,

$$L(\tau, \operatorname{ad}\tilde{P}) := \sum_{i=0}^{n} (-1)^i \operatorname{tr}(\tau^* | H^i(\tilde{M}, \operatorname{ad}\tilde{P})).$$

In our case, since τ acts on $\tilde{M}$ without fixed points, we get $L(\tau, \operatorname{ad}\tilde{P}) = 0$ regardless of whether n is even or odd. These statements are consistent with Lemma 2.7.

It can be shown that a flat connection D is reductive if and only if the pullback $\tilde{D}$ is so [15]. On the other hand, D is simple if $b_0(M, \operatorname{ad}'P) = 0$, whereas $\tilde{D}$ is simple if $b_0(\tilde{M}, \operatorname{ad}\tilde{P}) = 0$. Clearly, D is simple if $\tilde{D}$ is so, but the converse is not true. Similarly, D is good if $\tilde{D}$ is so, but the converse is not true either. In addition to the requirement in the orientable case that the flat connection is good, the smoothness of $\mathcal{M}^{\operatorname{flat}}(P)$ at $[D]$ requires further that $H^2(M, \operatorname{ad}'P) = 0$, whereas that of $\mathcal{M}^{\operatorname{flat}}(\tilde{P})$ at $[\tilde{D}]$ requires $H^2(\tilde{M}, \operatorname{ad}'\tilde{P}) = 0$. By Lemma 2.7, the vanishing of $H^2(\tilde{M}, \operatorname{ad}'\tilde{P})$ implies that of $H^2(M, \operatorname{ad}'P)$, but the converse is not true. We refer the reader to the Appendix for various examples.

Proposition 2.8. *Suppose a flat connection D on $P \to M$ lifts to a good flat connection $\tilde{D}$ on $\tilde{P} \to \tilde{M}$. Then*
1. $\pi^*\colon \mathcal{M}^{\operatorname{flat}}(P) \to \mathcal{M}^{\operatorname{flat}}(\tilde{P})^\tau$ is a homeomorphism from a neighbourhood of $[D]$ to a neighbourhood of $[\tilde{D}]$.
2. if furthermore $H^2(\tilde{M}, \operatorname{ad}'P) = 0$, the above local homeomorphism is a local diffeomorphism, and $\dim_{\mathbb{C}}\mathcal{M}_\circ^{\operatorname{flat}}(P) = b_1^+(\tilde{M}, \operatorname{ad}\tilde{P}) = b_{n-1}^-(\tilde{M}, \operatorname{ad}\tilde{P})$.

Proof. 1. There is an induced τ-action on $\mathcal{S}(\tilde{D})$ such that $\mathcal{S}(\tilde{D})^\tau$ can be identified (via π^*) with $\mathcal{S}(D)$. The map $\mathcal{S}(\tilde{D}) \to \mathcal{A}^{\operatorname{flat}}(\tilde{P})/\mathcal{G}(\tilde{P})$ in Corollary 2.3 is τ-equivariant. Choose a sufficiently small τ-invariant neighbourhood $\tilde{\mathcal{V}}$ of $0 \in \mathcal{S}(\tilde{D})$. Then $\tilde{\mathcal{V}}^\tau$ is a neighbourhood of $0 \in \mathcal{S}(\tilde{P})^\tau$ that is homeomorphic to a neighbourhood of $[\tilde{D}] \in \mathcal{M}^{\operatorname{flat}}(P)^\tau$. On the other hand, $\tilde{\mathcal{V}}^\tau$ can be identified (via π^*) with a neighbourhood $\mathcal{V}$ of $0 \in \mathcal{S}(D)$ and is homeomorphic to a neighbourhood of $[D] \in \mathcal{M}^{\operatorname{flat}}(P)$.
2. This follows from Corollary 2.5. $\qquad\square$

Corollary 2.9. *Suppose P is a G-bundle over a compact non-orientable surface M homeomorphic to the connected sum of $h > 2$ copies of $\mathbb{R}\mathrm{P}^2$ and suppose D is a flat connection on P whose pullback $\tilde{D}$ to $\tilde{P} \to \tilde{M}$ is good. Then there is neighbourhood of $[D]$ in $\mathcal{M}_\circ^{\operatorname{flat}}(P)$ diffeomorphic to a neighbourhood of $\tilde{D}$ in $\mathcal{M}_\circ^{\operatorname{flat}}(\tilde{P})^\tau$. Moreover,*

$$\dim_{\mathbb{C}}\mathcal{M}_\circ^{\operatorname{flat}}(P) = \tfrac{1}{2}\dim_{\mathbb{C}}\mathcal{M}_\circ^{\operatorname{flat}}(\tilde{P}) = (h-2)\dim_{\mathbb{C}}G + \dim_{\mathbb{C}}Z(G).$$

Proof. Since $\tilde{M}$ is an orientable surface of genus $h-1$, we have, by Lemma 2.7,

$$\dim_{\mathbb{C}} H^1(\tilde{M}, \mathrm{ad}'\, \tilde{P})^\tau = \dim_{\mathbb{C}} H^1(\tilde{M}, \mathrm{ad}'\, \tilde{P})^{-\tau} = \tfrac{1}{2} \dim_{\mathbb{C}} H^1(\tilde{M}, \mathrm{ad}'\, \tilde{P}).$$

The results then follow easily from Proposition 2.8.2 and Corollary 2.6. $\qquad\square$

Finally, if $\tilde{M}$ is Kähler, then the action of τ on $\mathcal{M}_\circ^{\mathrm{flat}}(\tilde{P})$ is anti-symplectic with respect to ω in (2.5). Therefore, $\mathcal{M}_\circ^{\mathrm{flat}}(\tilde{P})^\tau$ is an isotropic submanifold in $\mathcal{M}_\circ^{\mathrm{flat}}(\tilde{P})$. If $\dim M = 2$, then $\mathcal{M}_\circ^{\mathrm{flat}}(\tilde{P})^\tau$ is a Lagrangian submanifold [14].

3. Algebraic approach to representation varieties

3.1. **Smooth points on the homomorphism space.** We assume that Π is a finitely generated group, i.e., $\Pi = F/N$, where $F = \langle X \rangle$ is the free group on a finite set $X := \{x_1, \ldots, x_d\}$ and N is a normal subgroup in F. An element $w \in F$ is a word in X, i.e., $w = \prod_{k=1}^l x_{i_k}^{m_k}$, where $m_k \in \mathbb{Z} \backslash \{0\}$ $(k = 1, \ldots, l)$. We also assume that Π is finitely presented, that is, in addition, N is the normal closure in F of a finite set $R := \{r_1, \ldots, r_q\} \subset N$. Each r_j $(j = 1, \ldots, q)$ is called a *relator*, and an element of N is of the form $\prod_{k=1}^m s_k r_{j_k}^{n_k} s_k^{-1}$, where $s_k \in F$, $1 \le j_k \le q$, $n_k \in \mathbb{Z} \backslash \{0\}$ $(k = 1, \ldots, m)$. Given a connected Lie group G, we have $\mathrm{Hom}(F, G) = G^X$ (the set of maps from X to G) since any homomorphism $\phi \colon F \to G$ is determined by its values on the generators, $(\phi(x_i))_{i=1,\ldots,d} \in G^X$. Each word $w = \prod_{k=1}^l x_{i_k}^{m_k} \in F$ defines a map $\tilde{w} \colon G^X \to G$, $(g_i)_{i=1,\ldots,d} \mapsto \prod_{k=1}^l g_{i_k}^{m_k}$. In particular, we have the maps $\tilde{r}_j \colon G^X \to G$ $(j = 1, \ldots, q)$ from the relators, and they form a single map $\tilde{\boldsymbol{r}} = (\tilde{r}_j)_{j=1,\ldots,q} \colon G^X \to G^R$. The space $\mathrm{Hom}(\Pi, G)$ can be identified with the subset $\tilde{\boldsymbol{r}}^{-1}(e, \ldots, e) = \bigcap_{j=1}^q \tilde{r}_j^{-1}(e)$ of G^X. We want to find a sufficient condition on $\phi \in \mathrm{Hom}(\Pi, G)$ so that the space $\mathrm{Hom}(\Pi, G)$ is smooth at ϕ.

Let $\mathfrak{g}$ be the Lie algebra of G. Composition of $\phi \in \mathrm{Hom}(F, G)$ with the adjoint representation of G on $\mathfrak{g}$ makes $\mathfrak{g}$ a $\mathbb{Z}F$-module, which we denote by $\mathfrak{g}_{\mathrm{Ad}\circ\phi}$. Recall that a 1-cocycles on F with coefficients in $\mathfrak{g}_{\mathrm{Ad}\circ\phi}$ is a map $\gamma \colon F \to \mathfrak{g}$ such that $\gamma(uv) = \gamma(u) + \mathrm{Ad}_{\phi(u)}\gamma(v)$ for all $u, v \in F$. The space $Z^1(F, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ of these 1-cocycles can be identified with $\mathfrak{g}^X$ as each 1-cocycle is determined by its values on the generators $x_1, \ldots, x_d$. On the other hand, by the left multiplication of G on G, the tangent space of $\mathrm{Hom}(F, G) = G^X$ at any point ϕ is also identified with $\mathfrak{g}^X$. If $\gamma = (\gamma_i)_{i=1,\ldots,d} \in \mathfrak{g}^X = T_\phi \mathrm{Hom}(F, G)$, let $\tilde{\gamma} \in Z^1(F, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ be the corresponding 1-cocycle satisfying $\tilde{\gamma}(x_i) = \gamma_i$ $(i = 1, \ldots, d)$. Then for any $\phi \in \mathrm{Hom}(F, G)$ and $w \in F$, $\gamma \in \mathfrak{g}^X$, we have [35] (see also [32, §VI])

$$(3.1) \qquad\qquad (d\tilde{w})_\phi(\gamma) = \tilde{\gamma}(w).$$

There is an action of Π on $\mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ because F acts on N by conjugation and on $\mathfrak{g}$ by $\mathrm{Ad} \circ \phi$, whereas N acts on $\mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ trivially. We denote the invariant subspace by $\mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi$. There is an evaluation map $\mathrm{ev}_R \colon \mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi \to \mathfrak{g}^R$, $\beta \mapsto (\beta(r_j))_{j=1,\ldots,q}$. The map ev_R is injective because if $\beta(R) = 0$, then $\beta(N) = 0$ by Π-invariance. So we can regard $\mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi$ as the subspace $\mathrm{im}(\mathrm{ev}_R)$ in $\mathfrak{g}^R$.

Theorem 3.1. *Let $(d\tilde{\boldsymbol{r}})_\phi \colon \mathfrak{g}^X \to \mathfrak{g}^R$ be the differential of the map $\tilde{\boldsymbol{r}} \colon G^X \to G^R$ at $\phi \in \mathrm{Hom}(\Pi, G)$. Then $\ker(d\tilde{\boldsymbol{r}})_\phi = Z^1(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ and $\mathrm{im}(d\tilde{\boldsymbol{r}})_\phi$ is contained in the subspace $\mathrm{im}(\mathrm{ev}_R)$ of $\mathfrak{g}^R$ that is isomorphic to $\mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi$, and*

$$(3.2) \qquad\qquad \mathrm{rank}(d\tilde{\boldsymbol{r}})_\phi = \dim \mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi - \dim H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}).$$

If $\mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi \cong \mathfrak{g}^R$, then $\mathrm{coker}(d\tilde{\boldsymbol{r}})_\phi \cong H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$.

Proof. Identifying $\mathrm{Hom}(\Pi, G)$ as a subset in G^X as above, the expected tangent space of $\mathrm{Hom}(\Pi, G)$ at $\phi \in \mathrm{Hom}(\Pi, G)$ is $\ker(d\tilde{\boldsymbol{r}})_\phi = \bigcap_{j=1}^q \ker(d\tilde{r}_j)_\phi \subset \mathfrak{g}^X = Z^1(F, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$. By (3.1), we obtain $\ker(d\tilde{\boldsymbol{r}})_\phi = \ker(\mathrm{res}_N^F)$, where $\mathrm{res}_N^F \colon Z^1(F, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) \to Z^1(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ is the restriction map. Therefore $\ker(d\tilde{\boldsymbol{r}})_\phi = Z^1(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$. Since N acts on $\mathfrak{g}_{\mathrm{Ad}\circ\phi} = \mathfrak{g}$ trivially, we have $Z^1(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) = \mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$. In fact, the image of res_N^F is contained in the invariant subspace $\mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi$. So we write $\mathrm{res}_N^F \colon Z^1(F, \mathfrak{g}_{\mathrm{Ad}\circ\rho}) \to \mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi$. By (3.1) again, we get a factorisation $(d\tilde{\boldsymbol{r}})_\phi = \mathrm{ev}_R \circ \mathrm{res}_N^F$ of the map $(d\tilde{\boldsymbol{r}})_\phi$ through $\mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi$, that is, the triangle in the diagramme

$$(3.3) \qquad \cdots \longrightarrow Z^1(F, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) \xrightarrow{\;\mathrm{res}_N^F\;} \mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi \xrightarrow{\;\Delta\;} H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) \longrightarrow 0$$

is commutative. Therefore we obtain

$$\mathrm{rank}(d\tilde{\boldsymbol{r}})_\phi = \mathrm{rank}(\mathrm{res}_N^F) = \dim \mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi - \dim \mathrm{coker}(\mathrm{res}_N^F).$$

If ev_R is surjective (hence an isomorphism), then $\mathrm{coker}(d\tilde{\boldsymbol{r}})_\phi \cong \mathrm{coker}(\mathrm{res}_N^F)$.

By a classical result of Eilenberg-MacLane [8], the horizonzal row in (3.3) is an exact sequence. There, $\Delta(\beta)$ is the obstruction to lifting $\beta \in \mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi$ to a 1-cocycle on F. More explicitly, let $c \colon \Pi \to F$ be any (set-theoretic) section and let $\tilde{c} \colon \Pi \times \Pi \to N$ be defined by $c(a_1)c(a_2) = c(a_1 a_2)\tilde{c}(a_1, a_2)$, where $a_1, a_2 \in \Pi$. Then

$\beta \circ \tilde{c} \in Z^2(\Pi, \mathfrak{g}_{\mathrm{Ado}\phi})$ and its class $\Delta(\beta) := [\beta \circ \tilde{c}] \in H^2(\Pi, \mathfrak{g}_{\mathrm{Ado}\phi})$ does not depend on the choice of c. This exact sequence also follows from the Lyndon-Hochschild-Serre spectral sequence [24, 16] of group cohomology associated to $\Pi = F/N$ or from Gruenberg's resolution [13]. Consequently, $\mathrm{coker}(\mathrm{res}_N^F) \cong H^2(\Pi, \mathfrak{g}_{\mathrm{Ado}\phi})$ and the results follow. $\qquad\square$

The vanishing of $H^2(\Pi, \mathfrak{g}_{\mathrm{Ado}\phi})$ is the condition that any vector in $Z^1(\Pi, \mathfrak{g}_{\mathrm{Ado}\phi})$ can be integrated to a curve in $\mathrm{Hom}(\Pi, G)$ (cf. [11, 12]). In the standard literature, this is also taken as the condition for ϕ to be a smooth point on $\mathrm{Hom}(\Pi, G)$. However, using the implicit function theorem, our result is that ϕ is a smooth point on $\mathrm{Hom}(\Pi, G)$ if $\mathrm{rank}(d\tilde{r})_\phi$ reaches the maximum value. Here $\mathrm{rank}(d\tilde{r})_\phi$ depends on both $H^2(\Pi, \mathfrak{g}_{\mathrm{Ado}\phi})$ and $\mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ado}\phi})^\Pi \cong \mathrm{im}(\mathrm{ev}_R)$ by (3.2). When the map ev_R is surjective, our condition reduces to the condition that $\dim H^2(\Pi, \mathfrak{g}_{\mathrm{Ado}\phi})$ is minimal, which is clearly satisfied if $H^2(\Pi, \mathfrak{g}_{\mathrm{Ado}\phi}) = 0$ or $H^2(\Pi, \mathfrak{g}'_{\mathrm{Ad}'\circ\phi}) = 0$.

Corollary 3.2. *Let $\Pi = F/N$ be a finitely presented group and G be a connected Lie group. If at $\phi \in \mathrm{Hom}(\Pi, G)$, the number $\dim \mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ado}\phi})^\Pi - \dim H^2(\Pi, \mathfrak{g}_{\mathrm{Ado}\phi})$ reaches its maximal value, then ϕ is a smooth point on $\mathrm{Hom}(\Pi, G)$ and $T_\phi \mathrm{Hom}(\Pi, G) \cong Z^1(\Pi, \mathfrak{g}_{\mathrm{Ado}\phi})$. The smooth part of $\mathrm{Hom}(\Pi, G)$ is of dimension*

$$|X| \dim G - \dim \mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ado}\phi})^\Pi + \dim H^2(\Pi, \mathfrak{g}_{\mathrm{Ado}\phi}).$$

If ev_R is surjective or equivalently, if $\mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ado}\phi})^\Pi \cong \mathfrak{g}^R$, then the dimension of the smooth part of $\mathrm{Hom}(\Pi, G)$ becomes $(|X| - |R|) \dim G + \dim H^2(\Pi, \mathfrak{g}_{\mathrm{Ado}\phi})$, where ϕ is chosen so that $H^2(\Pi, \mathfrak{g}_{\mathrm{Ado}\phi})$ is of minimal dimension. In general, ev_R is not surjective as the relators in the presentation might not be independent themselves. A *relation among the relators* is of the form $s(r_1, \ldots, r_q) = e$, where s is a word on R such that it becomes the identity element when each r_j $(j = 1, \ldots, q)$ is substituted by the word on X it represents. More precisely, let F_R be the free group generated by R and let $\varrho: F_R \to N \subset F$ be the homomorphism defined by elements of R as words on X. A relation among the relators is given by an element $s \in F_R$ such that $\varrho(s)$ is the identity element of N. Any relation among the relators, say s, defines a map $\tilde{s}: G^R \to G$ such that $\tilde{s} \circ \tilde{r}: G^X \to G$ is the constant map taking value of the identity element. Clearly, the image of $\tilde{r}$ is contained in $\tilde{s}^{-1}(e)$. We show that the subspace $\mathrm{im}(\mathrm{ev}_R) \subset \mathfrak{g}^R$, which is isomorphic to $\mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ado}\phi})^\Pi$, is contained in the formal tangent space of $\tilde{s}^{-1}(e)$ at $(e, \ldots, e) \in G^R$, and therefore it can not be the whole space $\mathfrak{g}^R$ if there are non-trivial relations among the relators in the presentation of Π.

Lemma 3.3. *If s is a relation among the relators, then $\mathrm{im}(\mathrm{ev}_R) \subset \ker(d\tilde{s})_{(e,\ldots,e)}$.*

Proof. Note that both N and F_R acts trivially on the module $\mathfrak{g}_{\mathrm{Ado}\phi}$. Under the identifications $Z^1(N, \mathfrak{g}_{\mathrm{Ado}\phi})^\Pi \cong \mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ado}\phi})^\Pi$ and $Z^1(F_R, \mathfrak{g}) \cong \mathfrak{g}^R$, the map $\mathrm{ev}_R: \mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ado}\phi})^\Pi \to \mathfrak{g}^R$ intertwines with the pullback map $\varrho^*: Z^1(N, \mathfrak{g}_{\mathrm{Ado}\phi})^\Pi \to Z^1(F_R, \mathfrak{g})$. So for any $\beta \in \mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ado}\phi})^\Pi$, we deduce from (3.1) that

$$d\tilde{s}(\mathrm{ev}_R(\beta)) = (\varrho^*\tilde{\beta})(s) = \tilde{\beta}(\varrho(s)) = \tilde{\beta}(e) = 0,$$

which verifies the result. $\qquad\square$

3.2. **Fox calculus, the relation module and the free resolution.** In this subsection, using additional algebraic concepts, we provide another derivation of Theorem 3.1, together with further understandings of the condition $\mathrm{Hom}(N, \mathfrak{g})^\Pi \cong \mathfrak{g}^R$ there.

Let w be a word in the free group F on the finite set $X = \{x_1, \ldots, x_d\}$ as in §3.1. If G is a Lie group, the differential (3.1) of the map $\tilde{w}: G^X \to G$ can also be expressed in terms of the Fox derivatives [10], which we now recall. A *derivation* on $\mathbb{Z}F$ is a $\mathbb{Z}$-linear map $\delta: \mathbb{Z}F \to \mathbb{Z}F$ such that $\delta(uv) = \delta(u)\epsilon(v) + u\delta(v)$ for all $u, v \in \mathbb{Z}F$, where $\epsilon: \mathbb{Z}F \to \mathbb{Z}$ is the *augmentation map*, i.e., $\epsilon(u) = \sum_{k=1}^m n_k$ if $u = \sum_{k=1}^m n_k w_k$, $w_k \in F$, $n_k \in \mathbb{Z}$. *Fox derivatives* ∂_i $(i = 1, \ldots, d)$ are derivations on $\mathbb{Z}F$ defined by $\partial_i(x_j) = \delta_{ij}$ $(i, j = 1, \ldots, d)$. The differential $d\tilde{w}$ in (3.1) can be expressed in terms of Fox derivatives. We have (cf. [11, 22]), for $\gamma = (\gamma_i)_{i=1,\ldots,d} \in \mathfrak{g}^X$,

$$(3.4) \qquad (d\tilde{w})_\phi(\gamma) = \sum_{i=1}^d \mathrm{Ad}_{\phi(\partial_i w)} \gamma_i.$$

The Abelianisation $\bar{N} := N/[N, N]$ of N is called the *relation module* of the presentation $\Pi = F/N$ [26]. It is a $\mathbb{Z}\Pi$-module: there is a Π-action on $\bar{N}$ because F acts on N by conjugation and hence on $\bar{N}$, while its subgroup N acts trivially on $\bar{N}$. Clearly, $\mathrm{Hom}(\bar{N}, \mathfrak{g}) = \mathrm{Hom}(N, \mathfrak{g})$ and $\mathrm{Hom}(\bar{N}, \mathfrak{g}_{\mathrm{Ado}\phi})^\Pi = \mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ado}\phi})^\Pi$.

There is a resolution of $\mathbb{Z}$ by free $\mathbb{Z}\Pi$-modules

$$(3.5) \qquad \cdots \xrightarrow{d_3} M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} \mathbb{Z},$$

where $M_0 = \mathbb{Z}\Pi$, $d_0 = \epsilon$ is the augmentation map, $M_1 = \bigoplus_{i=1}^d \mathbb{Z}\Pi \, \hat{x}_i$ has a basis $\{\hat{x}_i\}$ in 1-1 correspondence with X, $d_1(\hat{x}_i) = [x_i - 1]_\Pi$, $M_2 = \bigoplus_{j=1}^q \mathbb{Z}\Pi \, \hat{r}_j$ has a basis $\{\hat{r}_j\}$ in 1-1 correspondence with R, $d_2(\hat{r}_j) = \sum_{i=1}^d [\partial_i r_j]_\Pi \, \hat{x}_i$ (see for example [27, §II.3]). Here we denote by $[u]_\Pi$ the image of $u \in \mathbb{Z}F$ in $\mathbb{Z}\Pi$. It can be shown (see for example [17, §11.5, Theorem 1]) that $\ker(d_1) \cong \bar{N}$, the relation module.

The group cohomology $H^\bullet(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ is the cohomology of the cochain complex

$$\mathrm{Hom}_{\mathbb{Z}\Pi}(M_0, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) = \mathfrak{g} \xrightarrow{d_1^\vee} \mathrm{Hom}_{\mathbb{Z}\Pi}(M_1, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) = \mathfrak{g}^X \xrightarrow{d_2^\vee} \mathrm{Hom}_{\mathbb{Z}\Pi}(M_2, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) = \mathfrak{g}^R \xrightarrow{d_3^\vee} \cdots$$

dual to (3.5) with values in $\mathfrak{g}_{\mathrm{Ad}\circ\phi}$. The maps are $d_1^\vee = (\mathrm{Ad}_{\phi(x_i)} - 1)_{i=1,\ldots,d}$, $d_2^\vee(\gamma) = \left(\sum_{i=1}^d \mathrm{Ad}_{\phi(\partial_i r_j)}\gamma_i\right)_{j=1,\ldots,q}$ for $\gamma = (\gamma_i)_{i=1,\ldots,d} \in \mathfrak{g}^X$. By (3.4), we have $d_2^\vee = (d\tilde{r})_\phi$, and therefore

$$\mathrm{rank}(d\tilde{r})_\phi = \mathrm{rank}(d_2^\vee) = \dim\ker(d_3^\vee) - \dim H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}).$$

By the exact sequence $\cdots \to M_3 \xrightarrow{d_3} M_2 \xrightarrow{d_2} \ker(d_1) \to 0$, we get $\ker(d_3^\vee) \cong \mathrm{Hom}_{\mathbb{Z}\Pi}(\ker(d_1), \mathfrak{g}_{\mathrm{Ad}\circ\phi}) = \mathrm{Hom}(\bar{N}, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi$. Hence $\dim\ker(d_3^\vee) = \dim\mathrm{Hom}(\bar{N}, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi$, and (3.2) in Theorem 3.1 is recovered.

As remarked before, $\mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi \cong \mathfrak{g}^R$ is equivalent to the surjectivity of the map $\mathrm{ev}_R \colon \mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi \to \mathfrak{g}^R$. This is clearly satisfied if $M_3 = 0$ in the resolution (3.5), which implies that the cohomological dimension of Π is not greater than 2. In fact, M_3 should be thought of the module generated by the relations among the relators. In the special case when the presentation has a single relator [25], we show that ev_R is always surjective. Examples of groups whose presentations have a single relator are fundamental groups of compact orientable or non-orientable surfaces. Fox calculus was used in [11] to study the representation varieties of the fundamental groups of orientable surfaces; we will apply Fox calculus to the situation when the surfaces are non-orientable. See §3.4 for details.

Proposition 3.4. *Let Π be a group generated by a finite set X with a single relator r. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then r defines a map $\tilde{r} \colon G^X \to G$ and $\mathrm{Hom}(\Pi, G) = \tilde{r}^{-1}(e)$. If $\phi \in \mathrm{Hom}(\Pi, G)$, then $\mathrm{coker}(d\tilde{r})_\phi \cong H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$. If $\dim H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ is minimal at ϕ, then ϕ is a smooth point on $\mathrm{Hom}(\Pi, G)$ and $T_\phi \mathrm{Hom}(\Pi, G) \cong Z^1(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$. The smooth part of $\mathrm{Hom}(\Pi, G)$ is of dimension*

$$(|X| - 1)\dim G + \dim H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}).$$

Proof. We have $\Pi = F/N$, where F is the free group generated by X and N is the normal closure of $R := \{r\}$ in F. If r is not a proper power of any element in F, then $M_3 = 0$ in (3.5) and $\mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi \cong \mathfrak{g}$. If $r = s^m$ for some $s \in F$ and $m \geq 2$, then there is a short exact sequence $0 \to \mathbb{Z}\Pi([s]_\Pi - 1) \to \mathbb{Z}\Pi \to \bar{N} \to 0$ of $\mathbb{Z}\Pi$-modules (see for example [4, Proposition 2.4.19(b)]). Taking the dual sequence (with values in $\mathfrak{g}$), we obtain an exact sequence

$$0 \to \mathrm{Hom}(\bar{N}, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi \to \mathrm{Hom}_{\mathbb{Z}\Pi}(\mathbb{Z}\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) \to \mathrm{Hom}_{\mathbb{Z}\Pi}(\mathbb{Z}\Pi([s]_\Pi - 1), \mathfrak{g}_{\mathrm{Ad}\circ\phi}) \to \cdots.$$

We show that $\mathrm{Hom}_{\mathbb{Z}\Pi}(\mathbb{Z}\Pi([s]_\Pi - 1), \mathfrak{g}_{\mathrm{Ad}\circ\phi}) = 0$. In fact, if $f \in \mathrm{Hom}_{\mathbb{Z}\Pi}(\mathbb{Z}\Pi([s]_\Pi - 1), \mathfrak{g}_{\mathrm{Ad}\circ\phi})$, then for all $u \in \mathbb{Z}\Pi$, $f(u([s]_\Pi - 1)) = f(u[s]_\Pi \cdot 1) - f(u \cdot 1) = f(1) - f(1) = 0$ by Π-invariance. Consequently, $\mathrm{Hom}(\bar{N}, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi \cong \mathrm{Hom}_{\mathbb{Z}\Pi}(\mathbb{Z}\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) \cong \mathfrak{g}$. The results then follow from Corollary 3.2. $\qquad\square$

3.3. **Smooth points on the representation variety.** Let Π be a finitely generated group and G be a complex reductive Lie group with Lie algebra $\mathfrak{g}$. A homomorphism $\phi \in \mathrm{Hom}(\Pi, G)$ is *reductive* if the representation $\mathrm{Ad}\circ\phi$ of Π on $\mathfrak{g}$ is completely reducible; let $\mathrm{Hom}^{\mathrm{red}}(\Pi, G)$ be the set of reductive homomorphisms equipped with the subset topology. The action of G on $\mathrm{Hom}(\Pi, G)$ by conjugation on G preserves the subspace $\mathrm{Hom}^{\mathrm{red}}(\Pi, G)$. The quotient space $\mathcal{R}(\Pi, G) := \mathrm{Hom}^{\mathrm{red}}(\Pi, G)/G$ is the *representation variety* (or the *character variety*) of Π, and we denote by $[\phi]$ the point in $\mathcal{R}(\Pi, G)$ represented by $\phi \in \mathrm{Hom}^{\mathrm{red}}(\Pi, G)$. That ϕ is reductive is equivalent to the statement that the image $\phi(\Pi)$ is contained in the Levi subgroup of any parabolic subgroup containing $\phi(\Pi)$, or the condition that the orbit $G \cdot \phi$ is closed in $\mathrm{Hom}(\Pi, G)$. Therefore the quotient topology on $\mathcal{R}(\Pi, G)$ is Hausdorff. Whereas Corollary 3.2 and Proposition 3.4 give sufficient conditions for the smoothness of $\mathrm{Hom}(\Pi, G)$ at a point ϕ, the smoothness of the representation variety $\mathcal{R}(\Pi, G)$ at $[\phi]$ has further requirements due to possible quotient singularities. A crucial step of establishing the local structure of a quotient space like $\mathcal{R}(\Pi, G)$ is the slice theorem.

Suppose G acts algebraically on an affine variety X. Let $X/\!/G$ be the quotient by G of the subset of $x \in X$ such that $G \cdot x$ is closed, equipped with the quotient topology which is Hausdorff, and let $[x]$ be the point in $X/\!/G$ from x. A *slice* at $x \in X$ is a locally closed affine subvariety S containing x and invariant under the stabiliser G_x of x such that the G-action induces a G-equivariant homeomorphism from $G \times_{G_x} S$ to a neighbourhood of the orbit $G \cdot x$ in X. It is well known that a slice always exists for compact group actions. Luna's slice theorem [23] (see [7] for an introduction) is about the existence of a slice in the algebro-geometric context. A subset $U \subset X$ is *saturated* if for all $x \in U$ and $y \in X$, $\overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset$ implies $y \in U$. A saturated subset is G-invariant and a saturated open subset in X descends to an open subset in $X/\!/G$. Luna's slice theorem states that if $G \cdot x$ is closed in X, then there is a slice S at x such that the G-morphism $G \times_{G_x} S \to X$ induced by the action of G on X maps surjectively onto a saturated open subset $U \subset X$ and that the map $G \times_{G_x} S \to U$ is strongly étale. Recall from [7, Definition 4.14] that *strongly étale* implies that the induced map $S/\!/G \to U/\!/G$ is étale, which implies that the underlying analytic spaces are locally isomorphic in the complex topology [28, §III.5, Corollary 2]. Consequently, there is a homeomorphism between an open neighbourhood of $\{x\}/G_x$ in $S/\!/G_x$ and an open neighbourhood of $[x]$ in $X/\!/G$. If in addition X is smooth at x, then the slice S can be chosen smooth and we have $T_x X = T_x(G \cdot x) \oplus T_x S$. If furthermore G_x is

trivial or minimal at x, then $X/\!\!/G$ is smooth at $[x]$, and there is a diffeomorphism from a neighbourhood of $\{x\}/G_x$ in S/G_x to a neighbourhood of $[x]$ in $X/\!\!/G$.

Luna's slice theorem applies to our case because G is a complex reductive Lie group; it was applied to study the singularities of the character varieties of finitely generated free groups [9].

Theorem 3.5. *Let Π be a finitely presented group and G be a complex reductive Lie group. Then for any reductive homomorphism $\phi \in \mathrm{Hom}(\Pi, G)$, there is a slice S at ϕ of the G-action on $\mathrm{Hom}(\Pi, G)$ such that there is a G-equivariant local homeomorphism from $G \times_{G_\phi} S$ to a saturated open subset in $\mathrm{Hom}(\Pi, G)$. Consequently, there is a homeomorphism from a neighbourhood of $\{\phi\}/G_\phi$ in $S/\!\!/G_\phi$ to a neighbourhood of $[\phi]$ in $\mathcal{R}(\Pi, G)$. If in addition ϕ is a smooth point in $\mathrm{Hom}(\Pi, G)$, then S can be chosen smooth and $T_\phi \mathrm{Hom}(\Pi, G) = T_\phi(G \cdot \phi) \oplus T_\phi S$. If furthermore ϕ is good, then $[\phi]$ is a smooth point in $\mathcal{R}(\Pi, G)$, and there is a diffeomorphism from a neighbourhood of ϕ in S to a neighbourhood of $[\phi]$ in $\mathcal{R}(\Pi, G)$.*

Proof. The reductive group G is an affine variety and it acts algebraically on the affine variety $\mathrm{Hom}(F, G) = G^X$, where F is the free group on the finite set X of generators of Π. Since Π is finitely presented, $\mathrm{Hom}(\Pi, G)$ is an affine subvariety in G^X defined by finitely many relators in $R = \{r_j\}_{j=1,\dots q}$. The results then follow from Luna's slice theorem and its smooth version reviewed above. $\qquad\square$

The Lie algebra of the stabiliser subgroup G_ϕ is $\mathfrak{g}^\Pi = H^0(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$. Let Ad' be the adjoint action of G on $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ and let $\mathfrak{g}'_{\mathrm{Ad}'\circ\phi}$ be the vector space $\mathfrak{g}'$ equipped with the Π-module structure $\mathrm{Ad}' \circ \phi$. Since $H^i(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) = H^i(\Pi, \mathfrak{g}'_{\mathrm{Ad}'\circ\phi}) \oplus H^i(\Pi, Z(\mathfrak{g}))$, the dimension of $H^i(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ is minimal if $H^i(\Pi, \mathfrak{g}'_{\mathrm{Ad}'\circ\phi}) = 0$ for any $i \geq 0$. Let $\mathrm{Hom}_\circ(\Pi, G)$ be the set of $\phi \in \mathrm{Hom}^{\mathrm{red}}(\Pi, G)$ such that $G_\phi = Z(G)$ and the quantity $\dim_\mathbb{C} \mathrm{Hom}(N, \mathfrak{g})^\Pi - \dim_\mathbb{C} H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ reaches its maximum. Then $H^0(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) = Z(\mathfrak{g})$ and the quotient $\mathcal{R}_\circ(\Pi, G) := \mathrm{Hom}_\circ(\Pi, G)/G$ is in the smooth part of $\mathcal{R}(\Pi, G)$.

Corollary 3.6. *Suppose $\Pi = F/N$ is a finitely presented group, where F is a free group generated by a finite set X and N is a normal subgroup in F generated in F by a finite set of relators R. Let $\phi \in \mathrm{Hom}_\circ(\Pi, G)$. Then there is a neighbourhood of $[\phi]$ in $\mathcal{R}_\circ(\Pi, G)$ diffeomorphic to a neighbourhood of 0 in $H^1(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$, and*

$$\dim_\mathbb{C} \mathcal{R}_\circ(\Pi, G) = (|X| - 1)\dim_\mathbb{C} G + \dim_\mathbb{C} Z(\mathfrak{g}) - \dim_\mathbb{C} \mathrm{Hom}(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi + \dim_\mathbb{C} H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}).$$

In particular, if there exists $\phi \in \mathrm{Hom}_\circ(\Pi, G)$ such that $\mathrm{Hom}(N, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi \cong \mathfrak{g}^R$ and $H^2(\Pi, \mathfrak{g}'_{\mathrm{Ad}'\circ\phi}) = 0$, then

$$\dim_\mathbb{C} \mathcal{R}_\circ(\Pi, G) = (|X| - |R| - 1)\dim_\mathbb{C} G + \dim_\mathbb{C} Z(\mathfrak{g}) + \dim_\mathbb{C} H^2(\Pi, Z(\mathfrak{g})).$$

Proof. Recall that if we identify $\mathfrak{g}^X$ with $Z^1(F, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$, then $\ker(d\tilde{r})_\phi = Z^1(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$. Moreover, $T_\phi(G \cdot \phi) = B^1(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ because for any $\xi \in \mathfrak{g}$, the corresponding vector field on G^X at $(\phi(x_i))_{i=1,\dots,d}$ is $(\xi - \mathrm{Ad}_{\phi(x_i)}\xi)_{i=1,\dots,d}$, which is in $B^1(F, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$; it is in fact in $B^1(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ because $\phi(N) = 1$. The local diffeomorphism from $\mathcal{R}_\circ(\Pi, G)$ to $H^1(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ follows from $T_\phi S \cong Z^1(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})/B^1(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) = H^1(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$. The complex dimension of $\mathcal{R}_\circ(\Pi, G)$ is that of $H^1(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$; it is also equal to $\dim_\mathbb{C} \mathrm{Hom}_\circ(\Pi, G) - \dim_\mathbb{C}(G \cdot \phi)$, where $\dim_\mathbb{C} \mathrm{Hom}_\circ(\Pi, G)$ is given by Corollary 3.2 and $\dim_\mathbb{C}(G \cdot \phi) = \dim_\mathbb{C} G - \dim_\mathbb{C} Z(G)$. The dimension formulas then follow. $\qquad\square$

Let $\Pi = \pi_1(M)$ be the fundamental group of a compact Kähler manifold (M, ω) of complex dimension m. The universal cover $\overline{M} \to M$ is a principal Π-bundle and it is the pullback of the universal Π-bundle $E\Pi \to B\Pi$ by a classifying map $f : M \to B\Pi$ that induces an isomorphism on the fundamental groups. Recall that $H^k(\Pi, \mathbb{C}) \cong H^k(B\Pi, \mathbb{C})$ for any $k \geq 0$. Given $\phi \in \mathrm{Hom}(\Pi, G)$, for any $\alpha, \beta \in H^1(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$, consider the cup product $\alpha \cup \beta \in H^2(\Pi, \mathbb{C}) \cong H^2(B\Pi, \mathbb{C})$ defined using an invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$. From $[\omega] \in H^2(M, \mathbb{C})$, we get $[\omega^{m-1}/(m-1)!] \in H^{2(m-1)}(M, \mathbb{C})$ and its Poincaré dual $\mathrm{PD}[\omega^{m-1}/(m-1)!] \in H_2(M, \mathbb{C})$. There is a symplectic form ω_ϕ on $H^1(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ given by [19]

(3.6) $\qquad \omega_\phi(\alpha, \beta) = \langle \alpha \cup \beta, f_* \mathrm{PD}[\omega^{m-1}/(m-1)!] \rangle = \langle f^*(\alpha \cup \beta) \cup [\omega^{m-1}/(m-1)!], [M] \rangle.$

and it agrees with the gauge theoretic definition in (2.5). When M is a compact orientable surface (i.e.,$m = 1$), the symplectic form $\omega_\phi(\alpha, \beta) = \langle f^*(\alpha \cup \beta), [M] \rangle$ was constructed in [11].

3.4. **Representation variety of an index-2 subgroup.** Let Π be a finitely generated group containing a normal subgroup $\tilde{\Pi}$ of finite index. Then $\tilde{\Pi}$ is also finitely generated. If $\phi \in \mathrm{Hom}(\Pi, G)$, let $\tilde{\phi}$ be its restriction to $\tilde{\Pi}$. Then the finite group $\Gamma := \Pi/\tilde{\Pi}$ acts on $H^k(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})$ and $H^k(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})^\Gamma \cong H^k(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$. To verify this, we can use the Lyndon-Hochschild-Serre spectral sequence [24, 16], with $E_2^{pq} = H^p(\Gamma, H^q(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}}))$, converging to $H^k(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$. Since Γ is a finite group and $H^q(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})$ for all $q \geq 0$ are divisible, we have $E_2^{pq} = 0$ for all $p > 0, q \geq 0$, and thus $H^k(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) = E_2^{0k} = H^k(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})^\Gamma$ for all $k \geq 0$.

Though the general case is quite straightforward, we specialise to the case where $\tilde{\Pi}$ is an index-2 subgroup in Π. Since $\Gamma = \mathbb{Z}_2$, there is an involution τ on $H^k(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})$ and $H^k(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})^\tau \cong H^k(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ for all $k \geq 0$. The action of τ can be understood as follows. Pick any $c \in \Pi\backslash\tilde{\Pi}$. Then c acts on $\tilde{\Pi}$ by Ad_c and on $\mathfrak{g}$ by $\mathrm{Ad}_{\phi(c)}$. Let τ be the induced action of c on $H^k(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})$; it does not depend on the choice of c. Since $c^2 \in \tilde{\Pi}$ acts on the cohomology groups trivially, τ is an involution and, consequently,

$$H^k(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}}) = H^k(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})^\tau \oplus H^k(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})^{-\tau},$$

where $H^k(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})^{\pm\tau}$ are, respectively, the eigenspaces on which τ takes eigenvalues ± 1. As explained above, $H^k(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) \cong H^k(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})^\tau$.

If $\Pi = \pi_1(M)$ and $\tilde{\Pi} = \pi_1(\tilde{M})$, where M is a closed non-orientable surface and $\tilde{M}$ is its the oriented double cover, then $\tilde{\Pi}$ can be identified as an index-2 subgroup of Π. Choosing an invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$, there is a non-degenerate pairing, for each $k = 0, 1, 2$,

$$(3.7) \qquad H^k(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}}) \times H^{2-k}(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}}) \to H^2(\tilde{\Pi}, \mathbb{C}) \cong \mathbb{C}.$$

When $k = 1$, this is the symplectic form (3.6) on $H^1(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})$. Since the pairing (3.7) changes sign under τ, it induces non-degenerate pairings

$$H^k(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})^\tau \times H^{2-k}(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})^{-\tau} \to \mathbb{C}.$$

On the other hand, $H^1(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) \cong H^1(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})^\tau$ is a Lagrangian subspace in $H^1(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})$.

Another important consequence is that for the fundamental group Π of a non-orientable surface, $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ is not isomorphic to $H^0(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$. Indeed, (3.7) implies that $H^2(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})$ is isomorphic to $H^0(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}}) = \mathfrak{g}^{\tilde{\Pi}}$. Whereas $H^0(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) \cong H^0(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})^\tau = \mathfrak{g}^{\tilde{\Pi}} \cap \ker(\mathrm{Ad}_c - 1)$, we have $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) \cong H^2(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})^\tau \cong H^0(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})^{-\tau} = \mathfrak{g}^{\tilde{\Pi}} \cap \ker(\mathrm{Ad}_c + 1)$, and hence

$$(3.8) \qquad H^0(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}}) \cong H^2(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}}) \cong H^0(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) \oplus H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}).$$

Therefore $H^2(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}}) = 0$ implies $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) = 0$, but the converse is not true (see Appendix).

Suppose Π is the fundamental group of an orientable surface and $\phi \in \mathrm{Hom}(\Pi, G)$. In [11], Fox calculus was used to show that the rank of the map $(d\tilde{r})_\phi$ equals the codimension of centraliser of ϕ. In this case, the Lie algebra of the centraliser $H^0(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ is isomorphic to the second cohomology group $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$. If Π is the fundamental group of a non-orientable surface, we showed that the codimension of $\mathrm{im}(d\tilde{r})_\phi$ equals the dimension of $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ (Proposition 3.4), but $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ is not isomorphic to $H^0(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ as explained above. We can verify this in another way by giving an explicit formula for $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ using Fox calculus. If M is the connected sum of h copies of $\mathbb{R}P^2$, then its fundamental group $\Pi = \pi_1(M)$ is generated by $x_1, \ldots, x_h$ subject to one relation $\prod_{i=1}^h x_i^2 = e$.

Proposition 3.7. *Let Π be the fundamental group of a non-orientable surface as above and let G be a complex reductive Lie group. Then for any $\phi \in \mathrm{Hom}(\Pi, G)$, we have*

$$H^0(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) \cong \bigcap_{i=1}^h \ker(\mathrm{Ad}_{\phi(x_i)} - 1), \quad H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) \cong \bigcap_{i=1}^h \ker(\mathrm{Ad}_{\phi(x_i)} + 1).$$

Proof. The first equality is obvious because $H^0(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) = \mathfrak{g}^\Pi$. Since the presentation of Π has a single relator $r = \prod_{i=1}^h x_i^2$, we have, by Proposition 3.4, $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) \cong \mathrm{coker}(d\tilde{r})_\phi$, which can be calculated by Fox derivatives as follows. By (3.4), we obtain $\mathrm{im}(d\tilde{r})_\phi = \sum_{k=1}^h \mathrm{im}(\mathrm{Ad}_{\phi(\partial_k r)})$, where the Fox derivatives are $\partial_k r = \left(\prod_{i=1}^{k-1} x_i^2\right)(x_k + 1)$ for $1 \leq k \leq h$. Choose a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$. By the identity $\mathrm{im}(S(T+1))^\perp \cong \ker((T+1)S^{-1})$ for (complex) orthogonal transformations S and T on $\mathfrak{g}$, we get

$$\mathrm{coker}(d\tilde{r})_\phi \cong \mathrm{im}(d\tilde{r})_\phi^\perp = \bigcap_{k=1}^h \ker\left((\mathrm{Ad}_{\phi(x_k)} + 1) \prod_{i=1}^{k-1} \mathrm{Ad}_{\phi(x_{k-i})}^{-2}\right),$$

which agrees with our result on $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ upon a simplification. $\qquad \square$

The quotient map $\Pi \to \mathbb{Z}_2$ is given by sending any $\prod_{k=1}^l x_{i_k}^{m_k}$ (with $m_k \in \mathbb{Z}$) in Π to $\sum_{k=1}^l m_k$ (mod 2). Its kernel $\tilde{\Pi}$ is generated by $x_i x_j$ ($1 \leq i, j \leq h$), and we can choose any x_i as $c \in \Pi\backslash\tilde{\Pi}$. It is clear that

$$\bigcap_{i,j=1}^h \ker(\mathrm{Ad}_{\phi(x_i)\phi(x_j)} - 1) = \bigcap_{i=1}^h \ker(\mathrm{Ad}_{\phi(x_i)} - 1) \oplus \bigcap_{i=1}^h \ker(\mathrm{Ad}_{\phi(x_i)} + 1),$$

as subspaces of $\mathfrak{g}$, verifying the decomposition (3.8) using Fox calculus.

Proposition 3.7 provides an explicit formula for the second cohomology group $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ of the fundamental group Π of a non-orientable surface M. If we use other presentations of Π (for example, the one used in [14, 15]), similar explicit formulas for H^2 exist, also verifying (3.8). With the presentation here, if there is no non-zero vector $\xi \in \mathfrak{g}$ satisfying $\mathrm{Ad}_{\phi(x_i)}\xi = -\xi$ for all $1 \leq i \leq h$, then ϕ is a smooth point on $\mathrm{Hom}(\Pi, G) \subset G^h$, and $T_\phi \mathrm{Hom}(\Pi, G) \cong \mathfrak{g}^{h-1}$. If furthermore the stabiliser of ϕ is $Z(G)$, then $[\phi]$ is in the smooth part $\mathcal{R}_\circ(\Pi, G)$ of the representation variety, whose complex dimension is $(h-2)\dim_{\mathbb{C}} G + \dim_{\mathbb{C}} Z(G)$. Since the double cover $\tilde{M}$ has genus $h-1$, the complex dimension of $\mathcal{R}_\circ(\Pi, G)$ is half of that of $\mathcal{R}_\circ(\tilde{\Pi}, G)$. This agrees with the statement that $H^1(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ is a Lagrangian subspace of $H^1(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})$.

4. Relation of the two approaches

Suppose $\tilde{M} \to M$ is a regular covering and Γ is the group of deck transformations. Let $\Pi = \pi_1(M)$ and $\tilde{\Pi} = \pi_1(\tilde{M})$ as before. Then there is a fibration $E\Gamma \times_\Gamma \tilde{M} \to B\Gamma$ whose fibre is $\tilde{M}$ and whose total space, being the Borel construction of $\tilde{M}$ with the free Γ-action, is homotopic to $\tilde{M}/\Gamma = M$. Let G be a complex reductive Lie group and let $P \to M$ be a principal G-bundle with a flat connection. Then there is a G-bundle on the total space $E\Gamma \times_\Gamma \tilde{M}$ whose restriction to the fibre $\tilde{M}$ is the pull-back $\tilde{P} \to \tilde{M}$ of P. There is a spectral sequence with $\mathcal{E}_2^{pq} = H^p(\Gamma, H^q(\tilde{M}, \mathrm{ad}\,\tilde{P}))$ converging to $H^k(M, \mathrm{ad}\,P)$. This is called the Cartan-Leray spectral sequence associated to the regular covering $\tilde{M} \to M = \tilde{M}/\Pi$ [3, §VII.7], and it is the gauge theoretic analog of the Lyndon-Hochschild-Serre spectral sequence in group cohomology associated to $1 \to \tilde{\Pi} \to \Pi \to \Gamma \to 1$ with $E_2^{pq} = H^p(\Gamma, H^q(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\phi}))$, converging to $H^k(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ (cf. §3.4). Since $H^0(\tilde{M}, \mathrm{ad}\,\tilde{P}) = H^0(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})$, we have $\mathcal{E}_2^{k0} = E_2^{k0}$ for all $k \geq 0$.

As in §3.4, we apply the result to the case where M is a compact non-orientable manifold and $\tilde{M}$ is its orientation double cover. Then $\Gamma = \mathbb{Z}_2$ and the non-trivial element in $\mathbb{Z}_2$ acts on the cohomology groups $H^k(\tilde{M}, \mathrm{ad}\,\tilde{P})$ by τ (cf. 2.4). Since $\mathcal{E}_2^{pq} = 0$ for $p > 0, q \geq 0$, we obtain $H^k(M, \mathrm{ad}\,P) = \mathcal{E}_2^{0k} = H^k(\tilde{M}, \mathrm{ad}\,\tilde{P})^\tau$ for all $k \geq 0$. This is the gauge theoretic counterpart of the statement $H^k(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) \cong H^k(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}})^\tau$ in §3.4. Taking $k = 1$, we conclude that the (formal) tangent space of the moduli space of flat G-connections on M is the $\mathbb{Z}_2$-invariant subspace of the tangent space to the moduli space of flat G-connections on $\tilde{M}$.

Another important example is when $\tilde{M}$ is the universal cover $\overline{M}$ of M. Then $\Pi = \Gamma$ and the pullback of P to $\overline{M}$ is topologically the product bundle $\overline{M} \times G$. We have $H^q(\overline{M}, \mathrm{ad}\,\tilde{P}) \cong H^q(\overline{M}) \otimes \mathfrak{g}_{\mathrm{Ad}\circ\phi}$, where the group Π acts on $H^q(\overline{M})$ by the pullback of deck transformations as well as on $\mathfrak{g}$ by the homomorphism $\phi \colon \Pi \to G$ corresponding to the flat connection. Since $H^1(\overline{M}) = 0$, we have $\mathcal{E}_2^{p1} = 0$ for all $p \geq 0$, and $H^1(M, \mathrm{ad}\,P) = \mathcal{E}_2^{10} = H^1(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$. This means that the (formal) tangent space of the moduli space of flat G-connections on M in the gauge theoretic approach (cf. §2.3) is identical to that of the representation variety in a more algebraic approach (cf. §3.3).

To find information on the second cohomology groups, we note that since $\mathcal{E}_\infty^{11} \subset \mathcal{E}_2^{11} = 0$, there is a short exact sequence $0 \to \mathcal{E}_\infty^{20} \to H^2(M, \mathrm{ad}\,P) \to \mathcal{E}_\infty^{02} \to 0$. Here $\mathcal{E}_\infty^{20} = \mathcal{E}_3^{20} = \mathrm{coker}(\mathcal{E}_2^{01} \to \mathcal{E}_2^{20}) = \mathcal{E}_2^{20} = H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ and $\mathcal{E}_\infty^{02} = \ker(\mathcal{E}_3^{02} \to \mathcal{E}_3^{30}) = \ker(\mathcal{E}_2^{02} \to \mathcal{E}_2^{30}) = \ker(H^2(\overline{M}, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi \to H^3(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}))$. By the Hurewicz theorem, we have $H_2(\overline{M}) \cong \pi_2(\overline{M}) \cong \pi_2(M)$, and the isomorphisms are equivariant under Π. Hence $H^2(\overline{M}, \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi \cong \mathrm{Hom}(H_2(\overline{M}), \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi \cong \mathrm{Hom}(\pi_2(M), \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi$. On the other hand, we have $\mathcal{E}_2^{30} \subset H^3(M, \mathrm{ad}\,P)$ and because of $\mathcal{E}_2^{11} = 0$ again, we have $\mathcal{E}_2^{3,0} = \mathrm{coker}(\mathcal{E}_2^{02} \to \mathcal{E}_2^{30})$. So there is an exact sequence (cf. [3, Exer. 6, page 175])

$$(4.1) \qquad 0 \to H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) \to H^2(M, \mathrm{ad}\,P) \to \mathrm{Hom}(\pi_2(M), \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi \to H^3(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) \to H^3(M, \mathrm{ad}\,P).$$

This means that the second cohomology groups $H^2(M, \mathrm{ad}\,P)$ and $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$ that appear as obstructions to smoothness in the two approaches (cf. §2.3 and §3.1) are not identical but differ by $\mathcal{E}_\infty^{02} = \ker(\mathrm{Hom}(\pi_2(M), \mathfrak{g}_{\mathrm{Ad}\circ\phi})^\Pi \to H^3(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}))$. However, if $\pi_2(M) = 0$, then $H^2(M, \mathrm{ad}\,P) \cong H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi})$. This is the case if the universal cover $\overline{M}$ is contractible, for example if M is an orientable surface of genus $g \geq 1$, a non-orientable surface which is the connected sum of $h \geq 2$ copies of $\mathbb{R}P^2$, a hyperbolic 3-manifold, or a Kähler manifold which is a ball quotient.

In §2.3, smoothness at a point of the (gauge-theoretic) moduli space requires, among other things, the condition $H^2(M, \mathrm{ad}'\,P) = 0$, which appears stronger than its algebraic version $H^2(\Pi, \mathfrak{g}'_{\mathrm{Ad}'\circ\phi}) = 0$ by (4.1) as explained above. On the other hand, by applying the implicit function theorem in §3.1 and §3.2, the character variety $\mathcal{R}(\Pi, G)$ is smooth at points $[\phi]$ where $H^2(\Pi, \mathfrak{g}'_{\mathrm{Ad}'\circ\phi}) = 0$ and ev_R is surjective. While the condition $H^2(\Pi, \mathfrak{g}'_{\mathrm{Ad}'\circ\phi}) = 0$ is common in the two approaches, it would be interesting to compare the additional requirements that differ. Fortunately, for orientable and non-orientable surfaces, these additional conditions happen to be vacuous, though for different reasons, and we arrive at the same condition $H^2(\Pi, \mathfrak{g}'_{\mathrm{Ad}'\circ\phi}) = 0$ from both methods.

Let G be a reductive Lie group. Suppose M is a non-orientable manifold and $\pi\colon \tilde{M} \to M$ is its orientation double cover. A flat G-connection D on M lifts to a flat connection $\tilde{D}$ on $\tilde{M}$. Alternatively, D determines a homomorphism $\phi\colon \Pi \to G$, where $\Pi = \pi_1(M)$. The homomorphism $\tilde{\phi}\colon \tilde{\Pi} \to G$ from $\tilde{D}$ is the restriction of ϕ to $\tilde{\Pi} = \pi_1(\tilde{M})$, which is an index-2 subgroup of Π. We compare various conditions on D (or ϕ) and $\tilde{D}$ (or $\tilde{\phi}$).

First, D (or ϕ) is *simple* if the Lie algebra of its stabiliser is that of $Z(G)$. The condition is equivalent to $H^0(M, \mathrm{ad}' P) = 0$ or $H^0(\Pi, \mathfrak{g}_{\mathrm{Ad}'\circ\phi}) = 0$. Clearly, D (or ϕ) is simple if $\tilde{D}$ (or $\tilde{\phi}$) is so, but the converse is not true. In the examples below, we utilise the standard Pauli matrices

$$\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} & -\sqrt{-1} \\ \sqrt{-1} & \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

satisfying $\sigma_i \sigma_j = -\sigma_j \sigma_i$ $(i \neq j)$ and $\sigma_i^2 = I_2$, $\det(\sigma_i) = -1$ $(i = 1, 2, 3)$.

Example A.1. Take $M = \mathbb{R}P^2 \# \mathbb{R}P^2$ or Π generated by x_1, x_2 subject to a relation $x_1^2 x_2^2 = 1$. Let $G = \mathrm{SL}(2, \mathbb{C})$ (whose centre is finite) and ϕ be defined by $\phi(x_1) = \sqrt{-1}\sigma_1$, $\phi(x_2) = \sqrt{-1}\sigma_2$. Then by Proposition 3.7, we obtain $H^0(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) = 0$ and $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) \cong \mathbb{C}\sigma_3$. On the other hand, by (3.8) or by a direct calculation, we obtain $H^0(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}}) \cong H^2(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}}) \cong \mathbb{C}\sigma_3$. So ϕ is simple but $\tilde{\phi}$ is not.

Following [33], we say that D (or ϕ) is *irreducible* if it is reductive and simple. In [15], we proved that D (or ϕ) is reductive if and only if $\tilde{D}$ (or $\tilde{\phi}$) is so. Thus if $\tilde{D}$ (or $\tilde{\phi}$) is irreducible, then so is D (or ϕ). The converse is not true because ϕ in Example A.1 is indeed reductive and hence irreducible, while $\tilde{\phi}$ is not irreducible.

Using (3.8), we also conclude that $H^2(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}'\circ\tilde{\phi}}) = 0$ implies $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}'\circ\phi}) = 0$. But the converse is not true either as shown in the following example.

Example A.2. Take the same M (or Π) and G as in Example A.1 but let ϕ be defined by

$$\phi(x_1) = \begin{pmatrix} \zeta_8 & \\ & \zeta_8^{-1} \end{pmatrix}, \quad \phi(x_2) = \begin{pmatrix} \zeta_8^{-1} & \\ & \zeta_8 \end{pmatrix},$$

where $\zeta_8 := e^{\pi\sqrt{-1}/4}$ is the 8th root of unity. Then $H^2(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) = 0$ whereas

$$H^2(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}}) \cong H^0(\tilde{\Pi}, \mathfrak{g}_{\mathrm{Ad}\circ\tilde{\phi}}) \cong H^0(\Pi, \mathfrak{g}_{\mathrm{Ad}\circ\phi}) \cong \mathbb{C}\sigma_3 \neq 0.$$

A flat G-connection D (or $\phi \in \mathrm{Hom}(\Pi, G)$) is *good* (cf. [18]) if it is reductive and its stabiliser is precisely $Z(G)$. If D (or ϕ) is good, it is irreducible, but the converse is not true. In Example A.3, we will encounter a case in which the stabiliser contains the centre $Z(G)$ as a proper subgroup of finite index. It is also clear that D (or ϕ) is good if $\tilde{D}$ (or $\tilde{\phi}$) is so. But the converse is not true either. In Example A.1, the Lie algebra of the stabiliser changes when D is pulled back to $\tilde{M}$ (or when ϕ is restricted to $\tilde{\Pi}$). We will construct an example in which the stabiliser changes even if its Lie algebra remains the same when D is pulled back to $\tilde{M}$. To achieve this, we need to choose a reductive group G that does not have property CI of [33], so that there exists an irreducible subgroup in G whose centraliser is a non-trivial finite extension of $Z(G)$.

Example A.3. We choose the simplest non-CI group $G = \mathrm{PSL}(2, \mathbb{C})$ and denote its elements by $\pm g$, where $g \in \mathrm{SL}(2, \mathbb{C})$. Its centre $Z(G)$ is trivial, but there is a finite Abelian subgroup $H := \{\pm I_2, \pm\sqrt{-1}\sigma_1, \pm\sqrt{-1}\sigma_2, \pm\sqrt{-1}\sigma_3\}$, isomorphic to the Klein 4-group, whose centraliser in G is H itself [33]. We let $M = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$ or Π be generated by x_1, x_2, x_3 subject to a relation $x_1^2 x_2^2 x_3^2 = 1$. Define ϕ by

$$\phi(x_1) = \pm\begin{pmatrix} \zeta_8 & \\ & \zeta_8^{-1} \end{pmatrix}, \quad \phi(x_2) = \pm\begin{pmatrix} \zeta_8^{-1} & \\ & \zeta_8 \end{pmatrix}, \quad \phi(x_3) = \pm\begin{pmatrix} & -\zeta_8 \\ \zeta_8^{-1} & \end{pmatrix}.$$

Then the centraliser of $\phi(\Pi)$ in $G = \mathrm{PSL}(2, \mathbb{C})$ is trivial. Since $\tilde{\Pi}$ is generated by $x_i x_j$ $(1 \leq i, j \leq 3)$, it is easy to check that $\phi(\tilde{\Pi}) = H$ and hence the stabiliser of $\tilde{\phi} = \phi|_{\tilde{\Pi}}$ is H. So ϕ is good but $\tilde{\phi}$ is not.

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